

IMPLICIT EXTREMES AND IMPLICIT MAX-STABLE LAWS

HANS-PETER SCHEFFLER[†] AND STILIAN STOEV[‡]

ABSTRACT. Let X_1, \dots, X_n be iid random vectors and $f \geq 0$ be a non-negative function. Let also $k(n) = \operatorname{Argmax}_{i=1, \dots, n} f(X_i)$. We are interested in the distribution of $X_{k(n)}$ and their limit theorems. In other words, what is the distribution the random vector where a function of its components is extreme. This question is motivated by a kind of inverse problem where one wants to determine the extremal behavior of X when only explicitly observing $f(X)$. We shall refer to such types of results as to *implicit extremes*. It turns out that, as in the usual case of explicit extremes, all limit *implicit extreme value* laws are *implicit max-stable*. We characterize the regularly varying implicit max-stable laws in terms of their spectral and stochastic representations. We also establish the asymptotic behavior of *implicit order statistics* relative to a given homogeneous loss and conclude with several examples drawing connections to prior work involving regular variation on general cones.

1. INTRODUCTION

On January 21 in 1959, Ohio experienced an extreme flood, which was the most destructive such event since 1913 claiming 16 lives and \$100 million in damages. The root cause of this event was not entirely due to extreme precipitation. It was essentially due to *rain on frozen ground*, i.e., cold ground-freezing conditions combined with a rare case of moderately intensive rainfall due to a warm front [11]. In hydrology, it is well understood that floods are not simply caused by extreme precipitation but, in fact, involve a complex combination of factors including ground saturation, snow-melt, precipitation intensity, and duration. In such and many other applications *extreme loss* events are caused by unusual combination of factors, the marginal values of which may or may not be *extreme* but their coordinated effect is extreme. Such type of phenomena motivated us to focus on *extreme loss events* rather than extreme values and develop theory that helps understand and model the joint behavior of the factors leading to extreme losses.

More precisely, let $f : \mathbb{R}^d \rightarrow [0, \infty)$ be a non-negative function modeling the loss $f(x)$ associated with the values $x = (x_i)_{i=1}^d$ of d factors. Let also X be a random vector in \mathbb{R}^d , modeling the joint behavior of these d factors. Assuming that X_1, \dots, X_n are independent copies (measurements) of X , we are interested in the behavior of $X_{k(n)}$ leading to maximal loss. Namely, let

$$k(n) := \operatorname{Argmax}_{k=1, \dots, n} f(X_k),$$

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[†] University of Siegen, Germany, scheffler@mathematik.uni-siegen.de.

[‡] University of Michigan, Ann Arbor, USA, [ssstoev@umich.edu](mailto:sstoev@umich.edu).

where in the case of ties $k(n)$ is taken as the smallest index yielding the maximum.

In this paper, our main goal is to establish the asymptotic behavior of $X_{k(n)}$ under appropriate normalization. Operationally, $X_{k(n)}$ may be viewed as the *f-implicit maximum* of the X_k 's, i.e., the observation leading to maximal f -loss. As illustrated, the events leading to extreme losses $f(X)$ are of utmost importance in practice. Thus, given a loss functional of interest, the limit distribution of $X_{k(n)}$, as $n \rightarrow \infty$ provides a natural fundamental model for the joint dependence structure of the factors leading to such extremes.

Here, we focus on the case of *homogeneous* losses (see (3.15), below). In Theorem 3.14, under the assumption that X is *regularly varying* on the cone $\mathbb{R}^d \setminus \{f = 0\}$, we show that

$$(1.1) \quad \frac{1}{a_n} X_{k(n)} \implies Y, \quad \text{as } n \rightarrow \infty,$$

for some normalizing sequence $a(n) > 0$, where \implies denotes convergence in distribution.

The limit laws arising in (1.1) will be referred to as *f-implicit extreme value distributions*. As anticipated from the classic theory of (explicit) multivariate extremes, the limits in (1.1) have certain stability property with respect to the operation of *implicit maxima*. Indeed, if Y_k , $k = 1, \dots, n$ are independent copies of Y , then it turns out that, for all n , exists $a(n) > 0$ such that

$$(1.2) \quad \text{Argmax}_{Y_k, k=1, \dots, n} f(Y_k) \stackrel{d}{=} a(n)Y.$$

Random vectors satisfying (1.2) will be referred to as *f-implicit max-stable*. Our first result (Theorem 3.14) shows that all implicit extreme value laws are in fact implicit max-stable. The converse is also true. In Theorem 4.4, we characterize the implicit max-domain of attraction of all *f-implicit max-stable* laws associated with positive and continuous homogeneous loss functions f . It turns out that these laws are precisely the regularly varying distributions on the cone $\mathbb{R}^d \setminus \{f = 0\}$. This result shows that the generalized notion of regular variation on cones is a natural technical and conceptual approach to implicit extremes. The notion of regular variation on general cones originates from the works on hidden regular variation of Resnick and Maulik [14]. It is briefly defined and reviewed in Section 3.1 below from the perspective of generalized polar coordinates. More details and further applications or regular variation on cones can be found in the recent work of [13].

In Section 6, we discuss several examples that unveil connections to prior work by Ledford and Tawn [12], Draisma *et al* [7], and de Haan and Zhou [5]. The recent work of Dombry and Ribatet [6] on ℓ -Pareto processes involves very similar ideas to ours. It focuses on the limit behavior of a process X conditionally on the event that a certain loss functional $\ell(X)$ is extreme. In this sense, Dombry and Ribatet study implicit *exceedances* whereas we study implicit *maxima*. Technically, the two approaches: *implicit extremes* and *implicit exceedances* lead to different limits and contexts of application but they both have important virtues. Conceptually, implicit extremes correspond to the study of (implicit) maxima, while implicit exceedances to (implicit) peaks-over-threshold as defined by a suitable loss functional.

The paper is structured as follows. In Section 2, we start with the problem formulation and give a key technical lemma. In Section 3.1, we review regular variation on cones in \mathbb{R}^d using generalized polar coordinates. We provide a disintegration formula for the measure of regular variation via its spectral measure on a (generalized) unit sphere. In Section 3.2, we

establish limit theorems for implicit extremes under regular variation condition. We also give stochastic representations of the limit laws, where the disintegration formula plays an important role. The implicit maximum domain of attraction is characterized in Section 4. In Section 5, we define implicit order statistics relative to a given loss and establish their asymptotic behavior. We end with several examples and discuss related work in Section 6. Some technical proofs and auxiliary results are given in the Appendix.

2. PRELIMINARIES

Let $X_i = (X_i^{(1)}, \dots, X_i^{(d)})$ be iid \mathbb{R}^d -valued random vectors. Moreover, let $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be measurable. For $n \geq 1$ define

$$(2.1) \quad k(n) = \operatorname{argmax}\{f(X_1), \dots, f(X_n)\}$$

so that

$$(2.2) \quad f(X_{k(n)}) = \max\{f(X_1), \dots, f(X_n)\}.$$

In the case of ties, $k(n)$ is taken as the smallest index for which the maximum is attained.

We are interested in the distribution of $X_{k(n)}$ and their limit theorems. In other words, what is the joint distribution of the components of the random vector where a function of its components is extreme. Broadly speaking, this is motivated by a kind of inverse problem where one wants to determine the extremal behavior of X when only explicitly observing $f(X)$. This is why we shall refer to such types of results as to *implicit extremes*.

Lemma 2.1. *Let $G(y) := P(f(X) \leq y)$ be the distribution function of $f(X)$, where $X \stackrel{d}{=} X_1$. Then, for all measurable $A \subset \mathbb{R}^d$ we have*

$$(2.3) \quad n \int_A G(f(x)-)^{n-1} P_X(dx) \leq P\{X_{k(n)} \in A\} \leq n \int_A G(f(x))^{n-1} P_X(dx).$$

In particular, if G is continuous, then

$$(2.4) \quad P_{X_{k(n)}}(dx) = nG(f(x))^{n-1} P_X(dx) \equiv nP\{f(X) \leq f(x)\}^{n-1} P_X(dx).$$

Proof. We have that

$$(2.5) \quad \begin{aligned} P\{X_{k(n)} \in A\} &= \sum_{i=1}^n P\{X_i \in A, k(n) = i\} \\ &= \sum_{i=1}^n P\{X_i \in A, f(X_j) < f(X_i), 1 \leq j < i \text{ and } f(X_j) \leq f(X_i), i < j \leq n\}. \end{aligned}$$

Each term in the above sum is bounded above by

$$P\{X_i \in A, f(X_j) \leq f(X_i) \text{ for all } j \neq i\}.$$

Therefore, by using the fact that the X_i s are iid, we obtain

$$\begin{aligned} P\{X_{k(n)} \in A\} &\leq n \int_A P\{f(X_j) \leq f(x) \text{ for all } j \neq i | X_i = x\} P_X(dx) \\ &= n \int_A P\{f(X) \leq f(x)\}^{n-1} P_X(dx) \\ &= \int_A G(f(x))^{n-1} P_X(dx). \end{aligned}$$

This yields the upper bound in (2.3). Similarly, each term of the sum in the right-hand side of (2.5) is bounded below by

$$\begin{aligned} P\{X_i \in A, f(X_j) < f(X_i) \text{ for all } j \neq i\} &= \int_A P\{f(X) < f(x)\}^{n-1} P_X(dx) \\ &= \int_A G(f(x)-)^{n-1} P_X(dx), \end{aligned}$$

which completes the proof of (2.3). \square

3. IMPLICIT EXTREME VALUE LAWS

In this section, we establish limit theorems for $X_{k(n)}$ in (2.2). The emerging limits will be referred to as implicit extreme value distributions. To this end, we need to impose some assumptions on X and f . Since we are concerned with multivariate extreme value theory on \mathbb{R}^d , it is natural to work in the context of multivariate regular variation. We shall need, however, a slight extension, which considers this notion over general *sub-cones* of $\overline{\mathbb{R}}^d$.

3.1. Regular variation on cones. This exposition is motivated by the fundamental concept of *hidden regular variation* pioneered by Resnick and Maulik [14] (see also p. 324 in [15]). The recent work of [13] develops abstract and far-reaching theory in the context of metric spaces. The following presentation is tailored to our needs.

Let $\overline{\mathbb{R}}$ denote the extended Real line $[-\infty, \infty]$. The topology in $\overline{\mathbb{R}}$ is generated by the usual class of open sets in \mathbb{R} along with the open neighborhoods of $\pm\infty$ of the type $(a, \infty]$ and $[-\infty, a)$, $a \in \mathbb{R}$. Thus $\overline{\mathbb{R}}$ becomes compact.

Let also $\overline{\mathbb{R}}^d$ be the Cartesian d -power of the extended Real line, equipped with the product topology. The space $\overline{\mathbb{R}}^d$ is compact by Tichonoff's theorem. It is also separable and complete with respect to the metric

$$(3.1) \quad \rho(x, y) := \sum_{i=1}^d r(x_i, y_i), \quad x = (x_i)_{i=1}^d, y = (y_i)_{i=1}^d \in \overline{\mathbb{R}}^d,$$

where $r(x, y) := |\operatorname{atan}(x) - \operatorname{atan}(y)|$, where $\operatorname{atan}(\pm\infty) := \pm\pi/2$. In fact, $\overline{\mathbb{R}}^d$ is homeomorphic to the compact interval $[-\pi/2, \pi/2]^d$ equipped with the usual topology, where the map $x \mapsto \operatorname{atan}(x)$ taken coordinate-wise is one homeomorphism, for example.

The classic notion of multivariate regular variation involves the ‘punctured space’ $\overline{\mathbb{R}}^d \setminus \{0\}$. In our context, it is convenient to remove an entire cone rather than just the origin. Recall that $D \subset \overline{\mathbb{R}}^d$ is said to be a (positive) cone, if $\lambda D \subset D$, for all $\lambda > 0$.

Let D be a *closed positive cone* and consider the punctured space $\overline{\mathbb{R}}_D^d := \overline{\mathbb{R}}^d \setminus D$, equipped with the relative topology. As expected, we have the following characterization of compacts.

Fact 3.1. *A set $F \subset \overline{\mathbb{R}}_D^d$ is compact if and only if it is closed and bounded away from D , that is, $F \subset \overline{\mathbb{R}}^d \setminus U$, where $U \supset D$ is an open neighborhood of D in $\overline{\mathbb{R}}^d$.*

The proof is given in the appendix. We equip $\overline{\mathbb{R}}_D^d$ with the Borel σ -algebra generated by all open sets. We shall consider Radon measures on $\overline{\mathbb{R}}_D^d$, i.e. those that are finite on all compacts. Since $\overline{\mathbb{R}}_D^d$ can be represented as a countable union of compacts, the Radon measures are σ -finite. Recall that the Radon measures ν_n converge vaguely to another measure ν , written

$$\nu_n \xrightarrow{v} \nu, \quad \text{as } n \rightarrow \infty,$$

if and only if $\int h d\nu_n \rightarrow \int h d\nu$, $n \rightarrow \infty$, for all continuous $h : \overline{\mathbb{R}}_D^d \rightarrow \mathbb{R}$ that vanish outside some compact set in $\overline{\mathbb{R}}_D^d$. The limit ν is necessarily Radon.

Definition 3.2. Let D be a closed cone in $\overline{\mathbb{R}}^d$. A random vector X in \mathbb{R}^d is said to be regularly varying on $\mathbb{R}_D^d := \mathbb{R}^d \setminus D$ with exponent $\alpha > 0$, if there exists a non-trivial Radon measure ν on $\overline{\mathbb{R}}_D^d$ supported on \mathbb{R}_D^d and a regularly varying sequence $a_n > 0$ with exponent $1/\alpha$, such that

$$(3.2) \quad nP(a_n^{-1}X \in \cdot) \xrightarrow{v} \nu, \quad \text{as } n \rightarrow \infty.$$

In this case, we write $X \in RV_\alpha(\{a_n\}, D, \nu)$ or sometimes simply $X \in RV_\alpha(D, \nu)$.

Observe that the vague convergence in (3.2) involves measures defined on $\overline{\mathbb{R}}_D^d$ that vanish on the set of infinite points $\overline{\mathbb{R}}_D^d \setminus \mathbb{R}^d$. In the case when the *exceptional cone* is $D = \{0\}$, one recovers the usual notion of multivariate regular variation.

Remark 3.3. The above definition is closely related and in fact inspired by the fundamental concept of *hidden regular variation* of Resnick and Maulik (see e.g. p. 324 in [15]). Our definition, however, does not involve multiple cones and it does not require, in particular, that X be multivariate regularly varying on $\mathbb{R}_{\{0\}}^d$. In this sense, regular variation on cones is both more basic and less restrictive than hidden regular variation. For a general treatment and several equivalents to the above Definition 3.2, see [13] and also Proposition 3.9 below.

The limit ν in (3.2) has the scaling property

$$(3.3) \quad \nu(\lambda \cdot) = \lambda^{-\alpha} \nu(\cdot), \quad \text{for all } \lambda > 0$$

(see e.g. Theorem 3.1 in [13]). As in the classical case, (3.3) yields a *disintegration formula* for ν involving radial and angular (or spectral) components. Special care needs to be taken, however, in defining the unit sphere. One may take as the unit sphere the boundary of any star-shaped domain containing D in its interior (in $\overline{\mathbb{R}}^d$). In practice, however, it is easier to derive it from suitable generalized polar coordinates as follows.

Definition 3.4 (polar coordinates in $\mathbb{R}^d \setminus D$). Let D be a closed cone in $\overline{\mathbb{R}}^d$. Let also $\tau : \overline{\mathbb{R}}^d \rightarrow [0, \infty]$ be a continuous function such that $\{\tau = 0\} = D$ and $\tau(x) < \infty$ for all

$x \in \mathbb{R}^d$. We shall assume also that τ is 1-homogeneous, that is, $\tau(\lambda x) = \lambda\tau(x)$ for all $\lambda > 0$ and $x \in \overline{\mathbb{R}}^d$. For $x \in \overline{\mathbb{R}}^d \setminus (D \cup \{\tau = \infty\}) \supset \mathbb{R}^d \setminus D$, its *polar coordinates* are defined as

$$(3.4) \quad (\tau, \theta) := (\tau(x), x/\tau(x)),$$

where τ is referred to as the radial and θ is the angular component of $x = \tau\theta$.

Now, fix some polar coordinates as in (3.4). The corresponding *unit sphere* is

$$\overline{S} := \{x \in \overline{\mathbb{R}}_D^d : \tau(x) = 1\}.$$

Since τ is continuous in $\overline{\mathbb{R}}_D^d$, the set \overline{S} is closed in $\overline{\mathbb{R}}_D^d$. It is also bounded away from $D \equiv \{\tau = 0\}$. Hence, the unit sphere \overline{S} is compact (in $\overline{\mathbb{R}}_D^d$). Consider the relative topology and corresponding Borel σ -algebra induced on \overline{S} . It can be shown that the map

$$(3.5) \quad T : \overline{\mathbb{R}}_D^d \setminus \{\tau = \infty\} \rightarrow (0, \infty) \times \overline{S},$$

defined as $T(x) := (\tau(x), \theta(x))$, is a homeomorphism of topological spaces. That is, T is one-to-one and onto, and both T and its inverse T^{-1} are continuous and hence measurable. The restriction of the map $T : \mathbb{R}^d \setminus D \rightarrow (0, \infty) \times S$, where

$$S := \overline{S} \cap \mathbb{R}^d \equiv \{\tau = 1\} \cap \mathbb{R}^d$$

is also a homeomorphism. The difference between \overline{S} and S is that the former may (and typically will) contain infinite points in $\overline{\mathbb{R}}^d \setminus \mathbb{R}^d$. Since the measures ν involved in (3.2) are supported on \mathbb{R}^d , however, we shall work with the uncompactified unit sphere S that contains only points from \mathbb{R}^d .

Any measure ν on $\mathbb{R}^d \setminus D$ naturally induces a measure $\tilde{\nu} := \nu \circ T^{-1}$ on $(0, \infty) \times S$, where $(0, \infty) \times S$ is equipped with the product σ -algebra. The disintegration formula for ν in (3.8) below is a consequence of the scaling property (3.3) and a change of variables. To gain intuition consider the ‘cylinder sets’

$$A_{r,B} = \{x : \tau(x) > r, \theta(x) \in B\} = T^{-1}((r, \infty) \times B), \quad r > 0, B \subset S$$

and observe that by the scaling property

$$(3.6) \quad \nu(A_{r,B}) = r^{-\alpha} \nu(A_{1,B}) = \tilde{\nu}((r, \infty) \times B).$$

This suggests defining the measure σ_S on S as follows

$$(3.7) \quad \sigma_S(B) := \nu(A_{1,B}) \equiv \tilde{\nu}((1, \infty) \times B), \quad B \subset S.$$

Note that $\sigma_S(S) = \nu\{\tau > 1\} < \infty$, since $\{\tau > 1\}$ is bounded away from D . By writing the term $r^{-\alpha}$ in (3.6) as $\int_r^\infty \alpha \tau^{-\alpha-1} d\tau$, we obtain the following result.

Fact 3.5. *Let ν be a Radon measure on $\overline{\mathbb{R}}_D^d$, supported on \mathbb{R}^d , which satisfies (3.3), for some $\alpha > 0$. Let also (τ, θ) be polar coordinates on $\overline{\mathbb{R}}_D^d$ as in (3.4). Then, there exists (unique) finite measure σ_S on $S := \{\tau = 1\} \cap \mathbb{R}^d$, such that*

$$(3.8) \quad \nu(A) = \int_S \int_0^\infty 1_A(\tau\theta) \frac{\alpha d\tau}{\tau^{\alpha+1}} \sigma_S(d\theta),$$

for all measurable $A \subset \overline{\mathbb{R}}_D^d$. The measure σ_S is uniquely identified by (3.7).

The measure σ_S in (3.8) will be referred to as a *spectral measure* of ν and will be used in the sequel to conveniently represent implicit extreme value laws. Depending on the cone D , the ‘right’ choice of polar coordinates and resulting unit ‘sphere’ may be somewhat counter-intuitive in applications as the following example shows. See also Example 3.1 in [13].

Example 3.6 (Pareto and Dirichlet). Let $X = (U_i^{-1/\alpha_i})_{i=1}^d$, where $U_i \stackrel{iid}{\sim} \text{Uniform}(0, 1)$ and $\alpha_i > 0$, $i = 1, \dots, d$. That is, the components of X are independent standard α_i -Pareto. It is well known that X is regularly varying in the usual sense, where the measure of regular variation ν concentrates on the coordinate axes corresponding to heaviest tail(s). More precisely, $X \in RV_\alpha(\{0\}, \nu)$, where $\alpha := \min_{i=1, \dots, d} \alpha_i$, and in this case

$$\nu\left(\bigtimes_{i=1}^d (x_i, \infty)\right) = \sum_{i=1}^d \delta_\alpha(\alpha_i) \frac{1}{x_i^{\alpha_i+1}}, \quad \text{for all } x_i > 0.$$

However, if one excises the axes as well as the origin, the random vector X becomes regularly varying with non-trivial measure ν supported on the entire positive orthant for all possible choices of positive exponents α_i .

Indeed, focus on the strictly positive orthant $\mathbb{R}_D^d := (0, \infty)^d$. Since

$$P(X^{(i)} > x_i, i = 1, \dots, d) = \prod_{i=1}^d x_i^{-\alpha_i} =: \nu\left(\bigtimes_{i=1}^d (x_i, \infty)\right), \quad \text{for all } x_i \geq 1,$$

it is easy to see that $X \in RV_\alpha(\{n^{1/\alpha}\}, D, \nu)$, where

$$(3.9) \quad \alpha = \sum_{i=1}^d \alpha_i, \quad \text{and} \quad \nu(dx_1 \cdots dx_d) = \prod_{i=1}^d \frac{\alpha_i}{x_i^{\alpha_i+1}} \times dx_1 \cdots dx_d.$$

We shall now determine the spectral measure of ν in suitable polar coordinates. Let

$$(3.10) \quad \tau(x) = \left(\sum_{i=1}^d \frac{1}{(x_i)_+} \right)^{-1} \equiv \|1/x_+\|_{\ell_1}^{-1} \quad \text{and} \quad \theta(x) = x/\tau(x).$$

Observe that $\tau : \mathbb{R}^d \rightarrow [0, \infty]$ is 1-homogeneous, continuous, $\tau(x) < \infty$, $x \in \mathbb{R}^d$, and $\{\tau > 0\} = (0, \infty]^d$. Therefore, (τ, θ) are valid polar coordinates in $(0, \infty)^d \subset \mathbb{R}^d \setminus \{D \cup \{\tau = \infty\}\}$. Note also that the unit sphere $S = \{\tau = 1\} \cap \mathbb{R}^d$ can be parameterized as follows:

$$S = \left\{ (1/u_i)_{i=1}^d : u_i \in (0, 1), \sum_{i=1}^d u_i = 1 \right\}.$$

That is, S is the image of the open unit simplex with respect to the coordinate-wise inversion operation $\mathcal{I}(u_1, \dots, u_d) = (1/u_1 \cdots 1/u_d)$. With this parameterization, we have that $x_i = \tau/u_i$, $i = 1, \dots, d$ and a standard computation of Jacobians yields

$$dx_1 \cdots dx_d = \tau^{d-1} \prod_{i=1}^d u_i^{-2} \times d\tau du_1 \cdots du_{d-1},$$

where the free variables are $\tau \in (0, \infty)$ and u_i , $i = 1, \dots, d-1$ with $u_i > 0$, $\sum_{i=1}^{d-1} u_i < 1$. For notational convenience we let $u_n := (1 - \sum_{i=1}^{d-1} u_i)$.

Now, for the measure ν in (3.9), we obtain

$$\begin{aligned} \nu(d\tau du_1 \cdots du_{d-1}) &= \prod_{i=1}^d \frac{\alpha_i u_i^{\alpha_i+1}}{\tau^{\alpha_i+1}} \times \tau^{d-1} \prod_{i=1}^d u_i^{-2} \times d\tau du_1 \cdots du_{d-1} \\ (3.11) \quad &= \frac{\alpha d \tau}{\tau^{\alpha+1}} \times \frac{\prod_{i=1}^d \alpha_i}{\alpha} u_1^{\alpha_1-1} \cdots u_d^{\alpha_d-1} du_1 \cdots du_{d-1} =: \frac{\alpha d \tau}{\tau^{\alpha+1}} \times \sigma_S(d\theta). \end{aligned}$$

This calculation shows an intriguing fact that the spectral measure σ_S in (3.11) is up to a constant the lift of a Dirichlet distribution on the unit simplex. That is, with $\mathcal{I}(x) = 1/x$, we have that

$$\sigma_S(B) = c_{\{\alpha_i\}} P(\mathcal{I}(\xi) \in B), \quad \text{where } \xi \sim \text{Dirichlet}(\alpha_1, \dots, \alpha_d),$$

and where $c_{\{\alpha_i\}} = (\prod_{i=1}^d \alpha_i \Gamma(\alpha_i)) / (\alpha \Gamma(\alpha))$.

This result can be used to efficiently simulate from implicit max-stable laws and in fact to characterize all such laws that have spectral measures absolutely continuous with respect to σ_S (see Example 6.1 and Proposition 6.2, below).

Remark 3.7. Other choices of polar coordinates are possible with the caveat that the unit ‘sphere’ needs to be bounded away from D . The typical choice of a unit sphere $S = \{\|x\| = 1\} \setminus D$, where $\|\cdot\|$ is some norm in \mathbb{R}^d would not have worked well in the previous example. Indeed, it could provide a disintegration formula of the type (3.8), but the resulting spectral measure will be infinite. This is because the set S is not bounded away from D .

Remark 3.8. For most cones D it is not possible to extend the homogeneous polar coordinates as a homeomorphism to the *entire* space $\overline{\mathbb{R}}^d \setminus D$ (including all points at infinity). Indeed, consider Example 3.6, where $\overline{\mathbb{R}}_D^d = (0, \infty]^d$ and (τ, θ) are as in (3.10). Then, $(\tau, \theta) : (0, \infty]^d \setminus \{(\infty, \dots, \infty)\} \rightarrow (0, \infty) \times \overline{S}$ is a homeomorphism. Since $\{\infty\} \times \overline{S}$ is not homeomorphic to the single point (∞, \dots, ∞) , however, the polar coordinates do not extend to $\overline{\mathbb{R}}_D^d$.

In the classic case of $\overline{\mathbb{R}}_{\{0\}}^d$, the coordinates $\tau(x) := \|x\|$, $\theta(x) := x/\|x\|$, extend by continuity to $\overline{\mathbb{R}}_{\{0\}}^d$, where $\|\cdot\|$ is an arbitrary norm in \mathbb{R}^d . This is perhaps the only case when $(\tau, \theta) : \overline{\mathbb{R}}_{\{0\}}^d \rightarrow (0, \infty] \times \overline{S}$ is a homeomorphism.

We give next a version of the well-known characterization of regular variation on \mathbb{R}_D^d in terms of generalized polar coordinates. The proof is given in the Appendix.

Proposition 3.9. *Let (τ, θ) be polar coordinates in \mathbb{R}_D^d as in Definition 3.4. Then $X \in RV_\alpha(\{a_n\}, D, \nu)$ if and only if, for some $C > 0$ and all $x > 0$*

$$(3.12) \quad nP(a_n^{-1}\tau(X) > x) \xrightarrow{n \rightarrow \infty} Cx^{-\alpha} \quad \text{and} \quad P(\theta(X) \in \cdot \mid \tau(X) > u) \xrightarrow[u \rightarrow \infty]{w} \sigma_0(\cdot),$$

where $\sigma_0(\cdot)$ is a probability measure on the unit sphere S . In this case, the spectral measure σ_S of ν and σ_0 are related as follows $\sigma_S = C\sigma_0$, where $C = \nu(\{\tau > 1\})$.

We give next an extension of the standard Breiman-type lemma, which provides a useful way of constructing regularly varying distributions on cones.

Lemma 3.10 (Breiman in polar coordinates). *Let $X := ZV$, where Z and V be independent and such that Z is a positive random variable and V takes values in the cone $\mathbb{R}^d \setminus D$. Then, the conditions*

$$P(Z > x) \sim x^{-\alpha}, \quad x \rightarrow \infty \quad \text{and} \quad E(\tau^\alpha(V)) < \infty,$$

imply that for all $x > 0$

$$nP(n^{-1/\alpha}\tau(X) > x) \xrightarrow[n \rightarrow \infty]{} E(\tau^\alpha(V))x^{-\alpha} \quad \text{and} \quad P(\theta(X) \in \cdot \mid \tau(X) > u) \xrightarrow[u \rightarrow \infty]{TV} \sigma_V(\cdot),$$

where the last convergence is in the sense of total variation norm and

$$\sigma_V(B) := \frac{1}{E\tau^\alpha(V)} \int_{\mathbb{R}^d} 1_B(\theta(v))\tau^\alpha(v)P_V(dv).$$

In particular, $X \in RV_\alpha(\{n^{1/\alpha}\}, D, \nu)$, where the spectral measure of ν is $\sigma_S(\cdot) = E(\tau^\alpha(V))\sigma_V(\cdot)$.

Proof. By the extension of Breiman's lemma given in Lemma 2.3 (2) of [4], we have

$$(3.13) \quad P(\tau(X) > u) = P(Z\tau(V) > u) \sim u^{-\alpha}E(\tau^\alpha(V)), \quad \text{as } u \rightarrow \infty.$$

Now, for all measurable $B \subset S \equiv \{\tau = 1\} \cap \mathbb{R}^d$, by homogeneity and independence

$$(3.14) \quad \begin{aligned} P(\theta(X) \in B \mid \tau(X) > u) &= \frac{P(\theta(V) \in B, Z\tau(V) > u)}{P(\tau(X) > u)} \\ &= \int_{\mathbb{R}^d} 1_B(\theta(v)) \frac{P(Z > u/\tau(v))}{P(\tau(X) > u)} P_V(dv) =: \int_{\mathbb{R}^d} 1_B(\theta(v)) h_u(v) P_V(dv), \end{aligned}$$

where P_V stands for the law of V .

By setting $B = S$, we see that h_u are probability densities. Further, by (3.13) and since $u^\alpha P(Z > u/c) \rightarrow c^\alpha$, as $u \rightarrow \infty$, for all $c > 0$, we get that

$$h_u(v) \longrightarrow h(v) := \frac{\tau^\alpha(v)}{E\tau^\alpha(V)}, \quad \text{as } u \rightarrow \infty,$$

where the convergence is valid for all v since $h_u(v) \equiv 0$, by convention when $\tau(v) = 0$. Note that h is also a probability density with respect to P_V . Thus, by the Scheffe-type Lemma A.3, we get that, as $u \rightarrow \infty$,

$$P(\theta(X) \in \cdot \mid \tau(X) > u) \xrightarrow[u \rightarrow \infty]{TV} \sigma_V(\cdot) := \int_V 1_{(\cdot)}(\theta(v)) \frac{\tau^\alpha(v)}{E\tau^\alpha(V)} P_V(dv).$$

This along with (3.13) thanks to Proposition 3.9, implies that $X = ZV \in RV_\alpha(\{n^{1/\alpha}\}, D, \nu)$, where the spectral measure of ν is $\sigma_S(\cdot) = E(\tau^\alpha(V))\sigma_V(\cdot)$. \square

3.2. Limit theorems for implicit extremes. We start by listing the assumptions on f and X .

Assumption $\text{RV}_\alpha(D, \nu)$. Let $D \subset \mathbb{R}^d$ be a closed cone in \mathbb{R}^d . We suppose that $X \in \text{RV}_\alpha(\{a_n\}, D, \nu)$, that is, X is regularly varying on $\mathbb{R}^d \setminus D$ with index $\alpha > 0$ and some Radon measure ν that does not charge infinite lines, i.e. $\nu(\mathbb{R}^d \setminus (\mathbb{R}^d \cup D)) = 0$.

Assumption H. Let $f : \mathbb{R}^d \rightarrow [0, \infty]$ be Borel measurable, such that $f(x) < \infty$, $x \in \mathbb{R}^d$, $f(0) = 0$, and 1-homogeneous, that is,

$$(3.15) \quad f(\lambda x) = \lambda f(x) \quad \text{for all } \lambda > 0 \text{ and } x \neq 0.$$

We shall use in the sequel the following two conditions relating f and ν .

Assumption F. For all $\epsilon > 0$, the set $\{f > \epsilon\}$ is bounded away from D . Furthermore, for all compact $K \subset \mathbb{R}_D^d$, we have

$$(3.16) \quad \inf_{x \in K} f(x) > 0.$$

Remark 3.11. Assumption F implies that $\{f = 0\} = D$. Indeed, $\{f > 0\} = \cup_{\epsilon > 0} \{f > \epsilon\} \subset \mathbb{R}^d \setminus D$ and hence $D \subset \{f = 0\}$. On the other hand for all $x \in \mathbb{R}^d \setminus D$, there exists a compact $K \subset \mathbb{R}^d \setminus D$ such that $x \in K$ and thus $f(x) > 0$, by (3.16). This shows that $\{f = 0\} \subset D$ and hence $\{f = 0\} = D$.

Assumption C. We have $\nu(\overline{\text{Disc}(f)}) = 0$, where $\overline{\text{Disc}(f)}$ denotes the closure in \mathbb{R}_D^d of the set of all discontinuity points of f .

We fix some polar coordinates as in Definition 3.4 so that the map $x \mapsto (\tau, \theta)$ is a homeomorphism between the spaces \mathbb{R}_D^d and $(0, \infty) \times S$. Recall the disintegration formula from Fact 3.5:

$$(3.17) \quad \nu(A) = \int_S \int_0^\infty 1_A(\tau\theta) \frac{\alpha d\tau}{\tau^{\alpha+1}} \sigma(d\theta),$$

where $\sigma = \sigma_S$ is the unique finite *spectral measure* of ν , relative to these polar coordinates.

Remark 3.12. By the homogeneity of the function f , we have

$$(3.18) \quad f(x) = \tau(x) f_0(\theta(x)),$$

where $f_0 : S \rightarrow (0, \infty)$ may be viewed as the *angular* part of f . By (3.17), Assumption C is equivalent to $\sigma(\overline{\text{Disc}(f_0)}) = 0$, where S is equipped with the relative topology. This means that the atoms of the spectral measure σ do not coincide with discontinuity points of the angular component f_0 .

Remark 3.13. Assumptions F and C are clearly fulfilled if $f : \mathbb{R}^d \rightarrow [0, \infty]$ is continuous and such that $D = \{f = 0\}$. In view of (3.18), however, interesting discontinuous homogeneous functions can be constructed that should be covered by a limiting theory. This motivates the more general Assumption C.

Theorem 3.14. *Assumptions $RV_\alpha(D, \nu)$, H , F and C imply*

$$(3.19) \quad a_n^{-1} X_{k(n)} \Longrightarrow Y \quad \text{as } n \rightarrow \infty$$

where Y is a random vector taking values in $\overline{\mathbb{R}}^d \setminus D$, with distribution

$$(3.20) \quad P_Y(dx) = e^{-Cf(x)^{-\alpha}} \nu(dx)$$

where

$$(3.21) \quad C := \nu(\{z : f(z) > 1\}) = \int_S f(\theta)^\alpha \sigma(d\theta) < \infty.$$

The random vector Y is proper, i.e. takes values in $\mathbb{R}^d \setminus D$ since by assumption $\nu(\overline{\mathbb{R}}^d \setminus (\mathbb{R}^d \cup D)) = 0$, that is, ν does not charge points on the infinite lines.

Proof. By the upper bound in (2.3) of Lemma 2.1, for any measurable set $A \subset \mathbb{R}^d$,

$$(3.22) \quad \begin{aligned} P(a_n^{-1} X_{k(n)} \in A) &\leq n \int_{a_n A} P(f(X) \leq f(x))^{n-1} P_X(dx) \\ &= \int_A \left(1 - \frac{nP(f(X) > f(a_n x))}{n}\right)^{n-1} nP_{a_n^{-1}X}(dx) =: \int_A h_n^+ d\nu_n, \end{aligned}$$

where $\nu_n(dx) := nP_{a_n^{-1}X}(dx)$.

Similarly, using the lower bound in (2.3) along with the established (3.22), we obtain

$$(3.23) \quad \int_A h_n^- d\nu_n \leq P(a_n^{-1} X_{k(n)} \in A) \leq \int_A h_n^+ d\nu_n.$$

where

$$(3.24) \quad h_n^-(x) := \left(1 - \frac{nP(f(X) \geq f(a_n x))}{n}\right)^{n-1}.$$

We will show that the two measures $h_n^\pm d\nu_n$ sandwiching the law of $a_n^{-1} X_{k(n)}$ in (3.23) converge to the same limit, which will ultimately yield (3.19). We will first present the intuition and then make the argument precise.

By the homogeneity of f and (3.2), as $n \rightarrow \infty$,

$$(3.25) \quad \begin{aligned} nP(f(X) > f(a_n x)) &\equiv nP(f(X) > a_n f(x)) \\ &= nP(a_n^{-1} X \in \{y : f(y) > f(x)\}) \\ &\longrightarrow \nu(\{y : f(y) > f(x)\}). \end{aligned}$$

This is true, provided $\{y : f(y) > f(x)\}$ is a ν -continuity set, which is bounded away from D . If this is the case, for h_n^+ in (3.22), we have

$$(3.26) \quad h_n^+(x) \longrightarrow h^+(x) := e^{-\nu(\{y : f(y) > f(x)\})} \equiv e^{-f(x)^{-\alpha} \nu(\{f > 1\})}, \quad \text{as } n \rightarrow \infty,$$

where in the last relation we used the homogeneity of f and the scaling property of ν . Under similar conditions, for h_n^- in (3.24), we obtain

$$(3.27) \quad h_n^-(x) \longrightarrow h^-(x) := e^{-\nu(\{y : f(y) \geq f(x)\})} \equiv e^{-f(x)^{-\alpha} \nu(\{f \geq 1\})}, \quad \text{as } n \rightarrow \infty.$$

The limit functions h^+ and h^- coincide. Indeed, by the homogeneity of f and the scaling property of ν , for all $c > 0$

$$\nu(\{f \geq c\}) - \nu(\{f > c\}) = \nu(\{f = c\}) = c^{-\alpha} \nu(\{f = 1\}).$$

The sets $\{f = c\}$, $c > 0$ are, however, disjoint. This, since $\{f > \epsilon\} = \cup_{c>\epsilon} \{f = c\}$ has finite ν -measure for all $\epsilon > 0$ (Assumption F), implies that $\nu(\{f = c\}) = c^{-\alpha} \nu(\{f = 1\}) = 0$, for all $c > 0$. Consequently, $\nu(\{f > 1\}) = \nu(\{f \geq 1\})$ and

$$(3.28) \quad h^+(x) = h^-(x) =: h(x) = e^{-Cf(x)^{-\alpha}}, \quad \text{for all } x \in \mathbb{R}^d \setminus D.$$

Recall that by Assumption $\text{RV}_\alpha(D, \nu)$, we have $\nu_n \xrightarrow{\nu} \nu$, $n \rightarrow \infty$. Hence, Relations (3.26), (3.27) and (3.28) suggest that the probability measures in (3.23) converge to the same measure $P_Y(dx) = h(x)\nu(dx)$. We will show this is indeed the case by using Lemmas A.1 and A.2, given in the Appendix.

Since $\nu_n \xrightarrow{\nu} \nu$, as $n \rightarrow \infty$, by Lemma A.1, the measures in the right-hand side of (3.23) converge to $h(x)\nu(dx)$, provided (3.26) holds uniformly in x over all compact subsets of $\overline{\mathbb{R}}_D^d$. This is true, if (3.25) is valid uniformly in x over K , for all compacts $K \subset \overline{\mathbb{R}}_D^d$. Note that by (3.16), the function $f(x)$ is uniformly bounded away from 0 over the compact K . Therefore, Lemma A.2 (iii) applied with $y := f(x)$ to (3.25) yields the desired uniform convergence. The argument showing that the left-hand side in (3.23) converges to $h(x)\nu(dx)$ is similar.

To complete the proof, it remains to show that, in the limit, no mass is lost at infinity, and the measure $P_Y(dx) = h(x)\nu(dx)$ given by (3.20) and (3.21) is a valid probability distribution on $\mathbb{R}^d \setminus D$. Note first that by assumption ν does not put any mass on the infinite hyperplanes, i.e. $\nu(\overline{\mathbb{R}}_D^d \setminus \mathbb{R}_D^d) = 0$. Thus, the support of the measure P_Y is confined to $\mathbb{R}^d \setminus D$.

Using the 1-homogeneity of f and the scaling property of ν , we obtain for all $x \in \mathbb{R}^d \setminus D$

$$\nu\{y : f(y) > f(x)\} = \nu\{z : f(z) > 1\}f(x)^{-\alpha} = Cf(x)^{-\alpha}.$$

This shows that $h(x) = e^{-\nu\{y:f(y)>f(x)\}} = e^{-Cf(x)^{-\alpha}}$. Next, we establish the second expression for C in (3.21). Using the disintegration formula (3.17), we get

$$\begin{aligned} C = \nu\{z : f(z) > 1\} &= \int_S \int_0^\infty 1_{\{z:f(z)>1\}}(\tau\theta) \frac{\alpha d\tau}{\tau^{\alpha+1}} \sigma(d\theta) \\ &= \int_S \int_{f(\theta)^{-1}}^\infty \frac{\alpha d\tau}{\tau^{\alpha+1}} \sigma(d\theta) \\ &= \int_S f(\theta)^\alpha \sigma(d\theta) < \infty. \end{aligned}$$

As a by-product of the above computation, we obtain that the last integral is finite since it equals $C \equiv \nu\{f > 1\} < \infty$, by Assumption F.

Finally, using (3.17) again, we verify that (3.20) integrates to one

$$\begin{aligned} \int_{\mathbb{R}^d \setminus D} e^{-Cf(x)^{-\alpha}} \nu(dx) &= \int_S \int_{(0,\infty)} e^{-Cf(\tau\theta)^{-\alpha}} \frac{\alpha d\tau}{\tau^{\alpha+1}} \sigma(d\theta) \\ &= \int_S \int_{(0,\infty)} e^{-C\tau^{-\alpha}f(\theta)^{-\alpha}} \frac{\alpha d\tau}{\tau^{\alpha+1}} \sigma(d\theta) \\ &= C^{-1} \int_S f(\theta)^\alpha \sigma(d\theta) \int_0^\infty e^{-u} du = 1, \end{aligned}$$

where in the last line, we used the change of variables $u := C\tau^{-\alpha}f(\theta)^{-\alpha}$ and the already established Relation (3.21). \square

Remark 3.15. Suppose that $X \in RV_\alpha(D, \nu)$ and let f be a *continuous* non-negative homogeneous function. The requirement that $D = \{f = 0\}$ following from Assumption F can be circumvented. Indeed, if $D \subset \tilde{D} := \{f = 0\}$, then $X \in RV_\alpha(\tilde{D}, \tilde{\nu})$, where $\tilde{\nu} := \nu|_{\mathbb{R}^d \setminus \tilde{D}}$ is the restriction of ν . Then, by the continuity of f , Assumptions F, H and C hold and hence (3.19) is valid over the restricted space $\mathbb{R}^d_{\tilde{D}}$.

It may happen, however, that $\text{supp}(\nu) \subset \tilde{D}$ and so trivially $\tilde{\nu} = 0$. As seen in Example 3.6, above, essentially different measure of regular variation may arise on the restricted cone $\mathbb{R}^d_{\tilde{D}}$. This shows that, in general, when focusing on f -implicit extremes, the natural domain of regular variation is $\mathbb{R}^d \setminus \{f = 0\}$. Finally, if $\{f = 0\} \subset D$, the above argument may fail since X may not be regularly varying in the larger cone $\mathbb{R}^d \setminus \{f = 0\}$.

Remark 3.16. It is important to note that Theorem 3.14 may fail for *continuous* $f : \mathbb{R}^d \rightarrow [0, \infty)$ that are, however, not continuous on the extended space \mathbb{R}^d . Indeed, consider for example the 1-homogeneous function $f(x_1, x_2) := \sqrt{x_1 x_2}$, $x_1, x_2 \geq 0$, defined as 0 elsewhere. Let also $X = (1/U_1, 1/U_2)$ be as in Example 3.6 above, where U_1 and U_2 are independent $\text{Uniform}(0, 1)$. We have that $X \in RV_\alpha(\{n^2\}, D, \nu)$, where $\alpha = 2$, $D := \mathbb{R}^2 \setminus (0, \infty]^2$, and

$$\nu(dx_1 dx_2) = x_1^{-2} x_2^{-2} dx_1 dx_2 \quad \text{on } (0, \infty)^2.$$

One may be tempted to conclude that (3.19) holds. Notice that for all $C > 0$, we have

$$\int_{(0,\infty)^2} e^{-Cf^{-\alpha}(x)} \nu(dx) = \int_0^\infty \int_0^\infty e^{-Cx_1^{-1}x_2^{-1}} x_1^{-2} x_2^{-2} dx_1 dx_2 = \int_0^\infty \int_0^\infty e^{-Cu_1 u_2} du_1 du_2 = \infty$$

and therefore (3.20) does not define a valid probability distribution.

Definition 3.17. The limits arising in (3.19) will be referred to as (f, ν) -implicit extreme value laws.

We have the following probabilistic representation. Recall that a random variable Z is said to be standard α -Fréchet ($\alpha > 0$), if $P(Z \leq x) = e^{-x^{-\alpha}}$, $x > 0$.

Proposition 3.18. *The random vector Y in $\mathbb{R}^d \setminus D$ has an (f, ν) -implicit extreme value law if and only if for some measurable $g : S \rightarrow [0, \infty)$ with $\int_S g^\alpha(\theta) \sigma(d\theta) = 1$,*

$$(3.29) \quad Y \stackrel{d}{=} Z \frac{\Theta}{g(\Theta)},$$

where Z standard α -Fréchet and Θ is an independent of Z random vector taking values in S and having distribution $\sigma_g(d\theta) := g^\alpha(\theta)\sigma(d\theta)$.

Moreover, the function g in (3.29) is unique, modulo σ -null sets and, in the context of Theorem 3.14, it is given by $g(\theta) = C^{-1/\alpha}f(\theta)$, $\theta \in S$. (Note that $P(g(\Theta) = 0) = 0$ and so (3.29) is well-defined.)

Proof. (\Rightarrow) Suppose first that Y is a (f, ν) -implicit extreme value, that is, (3.19) holds for some $\alpha > 0$ and 1-homogeneous function f . In view of the disintegration formula (3.17) of the measure ν , the law of Y in (3.20) has the following representation in polar coordinates

$$P_Y(d\tau\sigma(d\theta)) = e^{-Cf(\tau\theta)^{-\alpha}} \frac{\alpha d\tau}{\tau^{\alpha+1}} \sigma(d\theta).$$

Consider an arbitrary ‘rectangle’ in polar coordinates, i.e. $A_{r,B} := \{x \in \mathbb{R}^d \setminus D : \tau \leq r, \theta(x) \in B\}$, for $r > 0$ and a Borel set $B \subset S$. We have that

$$\begin{aligned} P(Y \in A_{r,B}) &= \int_S \int_0^\infty 1_{A_{r,B}}(\tau\theta) e^{-Cf(\tau\theta)^{-\alpha}} \frac{\alpha d\tau}{\tau^{\alpha+1}} \sigma(d\theta) \\ &= \int_B \int_0^r e^{-Cf(\theta)^{-\alpha}\tau^{-\alpha}} \frac{\alpha d\tau}{\tau^{\alpha+1}} \sigma(d\theta) \\ (3.30) \quad &= \int_B \int_{Cf(\theta)^{-\alpha}r^{-\alpha}}^\infty e^{-u} du C^{-1} f(\theta)^\alpha \sigma(d\theta), \end{aligned}$$

where in the second relation we used the homogeneity of f and in the last relation, we made the change of variables $u := Cf(\theta)^{-\alpha}\tau^{-\alpha}$. Note that the inner integral in (3.30) equals

$$e^{-Cf(\theta)^{-\alpha}r^{-\alpha}} = P(ZC^{1/\alpha}f(\theta)^{-1} \leq r),$$

for a standard α -Fréchet variable Z . We therefore obtain

$$(3.31) \quad P(Y \in A_{r,B}) \equiv P(\tau(Y) \leq r, \theta(Y) \in B) = \int_B P(ZC^{1/\alpha}f(\theta)^{-1} \leq r) \tilde{\sigma}(d\theta),$$

where $\tilde{\sigma}(d\theta) := C^{-1}f(\theta)^\alpha \sigma(d\theta)$. Observe that the choice of the constant C ensures that $\tilde{\sigma}(d\theta)$ is a probability distribution on S .

Suppose now that Θ is an independent of Z , S -valued random vector with probability distribution $\tilde{\sigma}$. Using the independence of Z and Θ , we see that the right-hand side of (3.31) equals $P(ZC^{1/\alpha}f(\Theta)^{-1} \in A_{r,B})$. This shows that the distributions of Y and $Z\Theta/g(\Theta)$ coincide on the class of sets $A_{r,B}$, $r > 0$, $B \in \mathcal{B}(S)$, where

$$g(\theta) := C^{-1/\alpha}f(\theta), \quad \theta \in S.$$

Since the latter class is a π -system, generating the Borel σ -algebra on $\mathbb{R}^d \setminus D$, the π - λ theorem shows that (3.29) holds.

(\Leftarrow) Conversely, for an arbitrary non-negative measurable function $g : S \rightarrow [0, \infty)$ with $\int_S g(\theta)^\alpha \sigma(d\theta) = 1$, let $f(x) := \tau(x)g(x/\tau(x)) \equiv \tau g(\theta)$ be a 1-homogeneous function. Consider the random vector

$$X := Z \frac{\Theta}{g(\Theta)},$$

where Z and Θ are independent with standard α -Fréchet and σ_g laws, respectively. Let (Z_i, Θ_i) , $1 \leq i \leq n$ be independent copies of (Z, Θ) . By homogeneity

$$f(X_i) = f\left(Z_i \frac{\Theta_i}{g(\Theta_i)}\right) = Z_i \frac{g(\Theta_i)}{g(\Theta_i)} = Z_i, \quad 1 \leq i \leq n.$$

That is, $f(X_i)$, $1 \leq i \leq n$ are iid α -Fréchet, that do not depend on the directions $\Theta_i = \theta(X_i)$ of the vectors X_i . Hence the random variable $k(n)$ in (2.1) is independent of Θ_i , $1 \leq i \leq n$ and

$$(3.32) \quad X_{k(n)} = Z_{k(n)} \frac{\Theta_{k(n)}}{g(\Theta_{k(n)})} \stackrel{d}{=} \left(\bigvee_{i=1}^n Z_i \right) \frac{\Theta_1}{g(\Theta_1)} \stackrel{d}{=} n^{1/\alpha} X,$$

where in the last relation we used the fact that $\bigvee_{i=1}^n Z_i \stackrel{d}{=} n^{1/\alpha} Z$. Relation (3.32) shows that (3.19) holds trivially in this case, where $a_n := n^{1/\alpha}$ and $Y \stackrel{d}{=} X$. That is, any Y as in (3.29) can be a limit in (3.19).

To complete the proof, it remains to show that the function g in (3.29) is unique. By letting $r \rightarrow \infty$ in (3.30), we see that

$$P(\theta(Y) \in B) = \int_B C^{-1} f(\theta)^\alpha \sigma(t\theta),$$

for all Borel $B \subset S$. This uniquely identifies g as $g(\theta) = C^{-1/\alpha} f(\theta)$, $\theta \in S$, modulo σ -null sets. \square

Remark 3.19. Observe that (2.1) remains unchanged if f is replaced by $\psi \circ f$, for any monotone strictly increasing function ψ . This shows that the result of Theorem 3.14 automatically extends to functions f such that $\psi^{-1} \circ f$ is 1-homogeneous and satisfies the assumptions of the theorem.

The following result shows that (f, ν) -implicit max-stable laws appearing in Theorem 3.14 are also in the class $RV_\alpha(D, \nu)$, as expected.

Corollary 3.20. *If Y is an (f, ν) -implicit extreme value random vector as in Theorem 3.14, then $Y \in RV_\alpha(\{n^{1/\alpha}\}, D, \nu)$. In fact, for all $x > 0$, we have*

$$(3.33) \quad nP(n^{-1/\alpha}\tau(Y) > x) \xrightarrow{n \rightarrow \infty} \sigma_S(S)x^{-\alpha} \quad \text{and} \quad P(\theta(Y) \in \cdot | \tau(Y) > u) \xrightarrow[u \rightarrow \infty]{TV} \sigma_0(\cdot),$$

where σ_S is as in (3.8) and $\sigma_0(\cdot) = \sigma_S(\cdot)/\sigma_S(S)$.

Proof. The result readily follows from the Breiman-type Lemma 3.10, above, applied to $X := ZV$, where $V := \Theta/g(\Theta)$. Note that now the law P_V of V is concentrated on the deformed unit sphere $\{\theta/g(\theta) : \theta \in S\}$. \square

Remark 3.21. If $X \in RV_\alpha(\{a_n\}, D, \nu)$, then $X \in RV_\alpha(c\{a_n\}, D, c^{-\alpha}\nu)$, for all $c > 0$. Thus, upon rescaling, we can always ensure that the spectral measure is a probability measure.

The next result shows the uniqueness of the stochastic representation of the implicit extreme value laws.

Corollary 3.22. *The representation in (3.29) is unique. More precisely, if (α, g, σ_g) and $(\tilde{\alpha}, \tilde{g}, \tilde{\sigma}_{\tilde{g}})$ are two triplets parameterizing the right-hand side therein, then $\alpha = \tilde{\alpha}$, $\sigma_S = \tilde{\sigma}_S$, and $g = \tilde{g} \pmod{\sigma_S}$.*

Proof. Suppose that

$$(3.34) \quad Y \stackrel{d}{=} Z \frac{\Theta}{g(\Theta)} \stackrel{d}{=} \tilde{Z} \frac{\tilde{\Theta}}{g(\tilde{\Theta})},$$

where the tilded quantities correspond to the stochastic representation as in (3.29) with parameters $(\tilde{\alpha}, \tilde{g}, \tilde{\sigma}_{\tilde{g}})$. By Corollary 3.20, we have $\alpha = \tilde{\alpha}$ and $\sigma_S = \tilde{\sigma}_S$. On the other hand, by (3.34),

$$\theta(Y) \stackrel{d}{=} \theta\left(Z \frac{\Theta}{g(\Theta)}\right) = \Theta \stackrel{d}{=} \tilde{\Theta},$$

and hence $\sigma_g = \tilde{\sigma}_{\tilde{g}}$, which yields $g = \tilde{g} \pmod{\sigma_S \equiv \tilde{\sigma}_S}$. \square

Remark 3.23. What happens with the stochastic representation in (3.29) under another set of polar coordinates (τ^*, θ^*) ? Let $S^* = \{\tau^* = 1\}$ and define the natural bijection $\lambda : S \rightarrow S^*$, where $\lambda(\theta) = \theta/\tau^*(\theta)$ is simply a rescaled version of the vector θ . Suppose that (3.29) holds and observe that by homogeneity,

$$(3.35) \quad Z \frac{\Theta}{g(\Theta)} = Z \frac{\Theta/\tau^*(\Theta)}{g(\Theta/\tau^*(\Theta))} =: Z \frac{\Theta^*}{g(\Theta^*)}, \quad \text{surely (not just almost surely).}$$

Observe that Z and $\Theta^* := \Theta/\tau^*(\Theta)$ are independent and Θ^* takes values in the new unit sphere S^* . The uniqueness of the stochastic representation (Corollary 3.22) then implies that the right-hand side (3.35) provides the stochastic representation of Y with respect to the new polar coordinates.

It is remarkable that the relationship between the two stochastic representations is deterministic. That is, the two involve the same α -Fréchet random variable and *the same* directional component vector $\Theta/g(\Theta) \equiv \Theta^*/g(\Theta^*)$. This shows that the representation in (3.29) does not depend on the choice of polar coordinates.

4. IMPLICIT MAX-STABLE LAWS AND THEIR DOMAINS OF ATTRACTION

Relation (3.32) in the proof of Proposition 3.18 suggests the following notion of *f-implicit max-stable* distributions.

Definition 4.1. An \mathbb{R}^d -valued random vector X is said to be implicit max-stable with respect to a homogeneous function f , or simply *f-implicit max-stable*, if for all n , there exist $a_n > 0$ such that

$$(4.1) \quad a_n^{-1} X_{k(n)} \stackrel{d}{=} X,$$

where $k(n)$ is as in (2.1), and X_i , $1 \leq i \leq n$ are independent copies of X .

By (3.32), all (f, ν) -implicit extreme value laws are also *f-implicit max-stable*. Under the mild additional assumption that f is continuous, the converse is also true, as shown next.

Theorem 4.2. Let $f : \overline{\mathbb{R}}^d \rightarrow [0, \infty]$ be non-negative, continuous and 1-homogeneous function such that $f(x) < \infty$, $x \in \mathbb{R}^d$. Then, a distribution is strictly *f-implicit max-stable* if and only if it is a (f, μ) -implicit extreme value distribution, where μ is supported on $\mathbb{R}^d \setminus \{f = 0\}$ and satisfies the scaling property $\mu(\lambda \cdot) = \lambda^{-\alpha} \mu(\cdot)$ for all $\lambda > 0$ and some $\alpha > 0$.

Proof. (\Leftarrow): By the continuity of f , the assumptions of Theorem 3.14 hold, and the claim follows from Relation (3.32) in the proof of Proposition 3.18.

(\Rightarrow): Assume that (4.1) holds. By the homogeneity of f and the definition of $k(n)$, Relation (4.1) implies

$$a_n^{-1} f(X_{k(n)}) = a_n^{-1} \max\{f(X_1), \dots, f(X_n)\} \stackrel{d}{=} f(X)$$

for all $n \geq 1$. Then, $f(X)$ is a max-stable random variable supported on $[0, \infty)$. Hence, by classical extreme value theory, we know that $f(X)$ has an α -Fréchet distribution and $a_n = n^{1/\alpha}$ for some $\alpha > 0$. Thus, there exists a constant $C > 0$ such that

$$P\{f(X) \leq x\} = e^{-Cx^{-\alpha}} \quad \text{for all } x > 0.$$

This implies in particular that $f(X)$ has continuous distribution and by Lemma 2.1,

$$P\{X_{k(n)} \in A\} = n \int_A e^{-(n-1)Cf(x)^{-\alpha}} P_X(dx).$$

Note also that $P_X\{f = 0\} = P\{f(X) = 0\} = 0$. Thus, the mass of P_X is concentrated on $\mathbb{R}^d \setminus D$, where $D := \{f = 0\}$. Without loss of generality, in the rest of the proof, we shall consider all measures over $\mathbb{R}^d \setminus D$.

By (4.1) with $a_n = n^{1/\alpha}$ and using the homogeneity of f , we obtain that for all $n \geq 1$

$$(4.2) \quad P\{X \in A\} = P\{n^{-1/\alpha} X_{k(n)} \in A\} = \int_A e^{-(1-n^{-1})Cf(x)^{-\alpha}} n \cdot P_{n^{-1/\alpha} X}(dx),$$

for all measurable $A \subset \mathbb{R}^d \setminus D$. We will show that this implies

$$(4.3) \quad \mu_n(dx) := n \cdot P_{n^{-1/\alpha} X}(dx) \xrightarrow{v} \mu(dx) \quad \text{as } n \rightarrow \infty$$

for some Radon measure μ on $\mathbb{R}^d \setminus D$.

Indeed, (4.2) means that $g_n(x) := e^{-(1-n^{-1})Cf(x)^{-\alpha}}$ is the Radon-Nikodym derivative of P_X with respect to μ_n . Since $f(x) > 0$, we have $g_n(x) > 0$, for all $x \in \mathbb{R}^d \setminus D$ and hence $\mu_n \ll P_X$. Thus, letting $h_n := g_n^{-1} \equiv d\mu_n/dP_X$, we obtain

$$n \cdot P_{n^{-1/\alpha} X}(A) \equiv \mu_n(A) = \int_A h_n(x) P_X(dx).$$

Observe that $h_n(x) = e^{(1-n^{-1})Cf(x)^{-\alpha}}$ converges to $h(x) := e^{Cf(x)^{-\alpha}}$, as $n \rightarrow \infty$, uniformly over all compacts in $\mathbb{R}^d \setminus D$. Therefore, by applying Lemma A.1 with μ_n , h_n and h as above to the trivial case $\nu_n \equiv \nu := P_X$, we obtain (4.3), where in fact

$$(4.4) \quad \mu(A) = \int_A h(x) P_X(dx) \equiv \int_A e^{Cf(x)^{-\alpha}} P_X(dx).$$

Relation (4.3) means that $X \in RV_\alpha(\{n^{1/\alpha}\}, D, \mu)$. Furthermore, since $f(x) > 0$, for all $x \in \mathbb{R}^d \setminus D$, Relation (4.4) is equivalent to

$$(4.5) \quad P\{X \in A\} = \int_A e^{-Cf(x)^{-\alpha}} \mu(dx),$$

for all Borel sets $A \subset \mathbb{R}^d \setminus D$, showing that X has a (f, μ) -extreme value law. Notice that as in the proof of Theorem 3.14, the constant C satisfies (3.21). \square

Definition 4.3. Fix $f : \overline{\mathbb{R}}^d \rightarrow [0, \infty]$ as in Theorem 4.2. We say that a random vector belongs to the f -implicit domain of attraction of a (necessarily) f -implicit max-stable random vector Y , if there exist $a_n > 0$ such that

$$(4.6) \quad a_n^{-1} X_{k(n)} \Longrightarrow Y \quad \text{as } n \rightarrow \infty$$

where $k(n)$ is as in (2.1) and X_1, \dots, X_n are i.i.d. as X . We write $X \in \text{DOA}_f(Y)$ in this case.

Theorem 4.4. Let $f : \overline{\mathbb{R}}^d \rightarrow [0, \infty]$ be non-negative, continuous and 1-homogeneous function, such that $f(x) < \infty$, $x \in \mathbb{R}^d$. Then, $X \in \text{DOA}_f(Y)$ if and only if $X \in \text{RV}_\alpha(\{f = 0\}, \mu)$ for some $\alpha > 0$.

Proof. (\Leftarrow): Theorem 3.14 shows that if $X \in \text{RV}_\alpha(\{f = 0\}, \nu)$, then $X \in \text{DOA}_f(Y)$ and Y has a f -implicit max-stable law by Theorem 4.2.

(\Rightarrow): Assume now that (4.6) holds. Then, by the continuous mapping theorem, we have

$$a_n^{-1} f(X_{k(n)}) = a_n^{-1} \max\{f(X_1), \dots, f(X_n)\} \Longrightarrow f(Y) \quad \text{as } n \rightarrow \infty.$$

This shows that $f(X)$ belongs to the domain of attraction of the (necessarily) α -Fréchet random variable $f(Y)$. Thus, a_n is regularly varying with index $1/\alpha$ and there exists a constant $C > 0$ such that

$$(4.7) \quad P\{a_n^{-1} f(X) \leq y\}^{n-1} \rightarrow e^{-Cy^{-\alpha}} \quad \text{as } n \rightarrow \infty$$

uniformly in $y > 0$. In view of Lemma 2.1, we then get

$$(4.8) \quad \int_A g_n^-(x) \mu_n(dx) \leq P\{a_n^{-1} X_{k(n)} \in A\} \leq \int_A g_n^+(x) \mu_n(dx),$$

where $\mu_n(\cdot) := nP(a_n^{-1} X \in \cdot)$, and where

$$g_n^-(x) = P\{a_n^{-1} f(X) < f(x)\}^{n-1} \quad \text{and} \quad g_n^+(x) = P\{a_n^{-1} f(X) \leq f(x)\}^{n-1}.$$

Notice that by (4.7),

$$(4.9) \quad g_n^\pm(x) \xrightarrow[n \rightarrow \infty]{} g(x) := e^{-Cf(x)^{-\alpha}}, \quad \text{for all } x \in \mathbb{R}^d \setminus D \equiv \mathbb{R}^d \setminus \{f = 0\}.$$

We will show that (4.8) and (4.9) imply

$$(4.10) \quad \mu_n(dx) \equiv n \cdot P_{a_n^{-1} X}(dx) \xrightarrow{v} \mu(dx), \quad \text{as } n \rightarrow \infty,$$

for some Radon measure μ on $\mathbb{R}^d \setminus \{f = 0\}$. To this end, observe that it is enough to show that for all fixed compacts $K \subset \overline{\mathbb{R}}^d \setminus \{f = 0\}$, we have

$$(4.11) \quad \mu_n(\cdot \cap K) \equiv n \cdot P_{a_n^{-1} X}(\cdot \cap K) \xrightarrow{w} \mu(\cdot \cap K), \quad \text{as } n \rightarrow \infty.$$

Proceeding as in the proof of Theorem 4.2, let $\nu_n(dx) := P\{a_n^{-1} X_{k(n)} \in dx\}$ and $\nu := P_Y$ be the laws of the left- and right-hand side in (4.6), respectively. Then, by (4.8), we have

$$(4.12) \quad g_n^-(x) \leq \frac{d\nu_n}{d\mu_n}(x) \leq g_n^+(x), \quad x \in \mathbb{R}^d \setminus \{f = 0\}.$$

The continuity of f over the compact K implies $\inf_{x \in K} f(x) > 0$. Thus, by Relation (4.9) for all sufficiently large n , we have $\inf_{x \in K} g_n^\pm(x) > 0$. This, in view of (4.12), shows that $\mu_n|_K \ll \nu_n|_K$, for all sufficiently large n and hence

$$\mu_n(A \cap K) = \int_{A \cap K} h_n(x) \nu_n(dx), \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{f = 0\}),$$

where

$$\frac{1}{g_n^+(x)} \leq h_n(x) := \frac{d\mu_n}{d\nu_n}(x) \leq \frac{1}{g_n^-(x)}, \quad x \in K.$$

By the uniformity of the convergence in (4.7) and the continuity of f , we also have that the convergences in (4.9) are uniform over the compact K . This shows that h_n converges to $h(x) := g^{-1}(x) = e^{Cf(x)^{-\alpha}}$, uniformly in $x \in K$, as $n \rightarrow \infty$. Thus, Lemma A.1 applied to the measures μ_n , ν_n and $\nu = P_Y$, restricted to K , yields (4.11). Since the choice of the compact K was arbitrary, we obtain (4.10), where

$$\mu(A) := \int_A h(x) \nu(dx) \equiv \int_A e^{Cf(x)^{-\alpha}} P_Y(dx).$$

Relation (4.10) and the fact that a_n is regularly varying with index $1/\alpha$ imply that $\mu(\lambda \cdot) = \lambda^{-\alpha} \mu(\cdot)$ for all $\lambda > 0$ and that $X \in RV_\alpha(\{a_n\}, \{f = 0\}, \mu)$. \square

Remark 4.5. In view of Corollary 3.20, f -implicit max-stable laws (for continuous f) are regularly varying and belong to their own domain of implicit attraction, as expected.

Remark 4.6. The continuity assumption in Theorem 4.2 can be relaxed. Note that the continuity of f is not used in the proof of the ‘only if’ part and it is only used in the ‘if’ part to justify the application of Theorem 3.14. Therefore, one can merely suppose that f satisfies the assumptions of the last theorem. The continuity assumption in Theorem 4.4 can be similarly relaxed.

5. IMPLICIT ORDER STATISTICS

In this brief section we study the natural counterpart of order statistics relative to a given loss function f . Namely, suppose that X_i , $i = 1, \dots, n$ are independent copies of a vector X . Consider the order statistics of the scalar sample of losses $\xi_i := f(X_i)$, $i = 1, \dots, n$. That is, let $\{k(1; n), \dots, k(n; n)\}$ be a permutation of $\{1, \dots, n\}$ such that

$$\xi_{k(1; n)} \equiv f(X_{k(1; n)}) \geq \xi_{k(2; n)} \equiv f(X_{k(2; n)}) \geq \dots \geq \xi_{k(n; n)} \equiv f(X_{k(n; n)}).$$

where, by convention, possible ties among the ξ_i ’s are resolved by taking the indices $k(\cdot; n)$ in an increasing order. We shall refer to $X_{k(i; n)}$, $i = 1, \dots, n$ as to the *implicit order statistics* relative to the loss f . Observe that $X_{k(1; n)} \equiv X_{k(n)}$ is the *implicit maximum* defined in (2.2) above.

We will establish the asymptotic behavior of the implicit order statistics for homogeneous losses and regularly varying X . To this end, it is convenient to consider polar coordinates generated by the loss function. Specifically, let $f : \overline{\mathbb{R}}^d \rightarrow [0, \infty]$ be a continuous homogeneous loss function, such that $f(x) < \infty$ for all $x \in \mathbb{R}^d$. Let also ν be a Radon measure on $\overline{\mathbb{R}}^d \setminus \{f = 0\}$, such that $\nu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ and

$$\nu(\lambda \cdot) = \lambda^{-\alpha} \nu(\cdot), \quad \text{for all } \lambda > 0,$$

with some exponent $\alpha > 0$.

Consider the *polar coordinates* $(\tau, \theta)(x) := (f(x), x/f(x))$, for $x \in \mathbb{R}^d \setminus \{f = 0\}$. By Fact 3.5, the measure ν satisfies the disintegration formula (3.8), with spectral measure

$$\sigma_S(B) := \nu((f, \theta) \in (1, \infty) \times B),$$

on the (finite) unit sphere $S = \{f = 1\} \cap \mathbb{R}^d$.

Theorem 5.1. *Let $f : \mathbb{R}^d \rightarrow [0, \infty]$ be a continuous homogeneous loss function, such that $f(x) < \infty$ for all $x \in \mathbb{R}^d$. Suppose that $X \in RV_\alpha(a_n, \{f = 0\}, \nu)$ and X_i , $i = 1, \dots, n$ are independent copies of X .*

(i) *Consider the Point process $\mathcal{N}_n := \{a_n^{-1}X_i, i = 1, \dots, n\}$. Then, as $n \rightarrow \infty$*

$$(5.1) \quad \mathcal{N}_n \cap \{f > 0\} \Longrightarrow \mathcal{N},$$

where \mathcal{N} is a Poisson process on $\mathbb{R}^d \setminus \{f = 0\}$ with intensity ν and ‘ \Rightarrow ’ denotes weak convergence of probability distributions on the space of random point measures equipped with the vague convergence topology.

(ii) *Moreover, with $c := \nu\{f > 1\} = \sigma_S(S)$, we have*

$$(5.2) \quad \mathcal{N} \stackrel{d}{=} c^{1/\alpha} \left\{ \Gamma_k^{-1/\alpha} \Theta_k, k \in \mathbb{N} \right\},$$

where $1 < \Gamma_1 < \Gamma_2 < \dots$ is a standard Poisson process on $(0, \infty)$. The Θ_k ’s are iid and independent of the Γ_k ’s random variables taking values on the unit sphere S and having distribution $\sigma_S(\cdot)/c$.

(iii) *In particular, for all $m \in \mathbb{N}$, as $n \rightarrow \infty$,*

$$(5.3) \quad \frac{1}{a_n}(X_{k(i;n)}, i = 1, \dots, m) \Longrightarrow c^{1/\alpha}(\Gamma_i^{-1/\alpha} \Theta_i, i = 1, \dots, m).$$

Proof. By Theorem 5.3 (i) on p. 138 in [15], the fact that $X \in RV_\alpha(\{a_n\}, D := \{f = 0\}, \nu)$ is equivalent to (5.1). This completes the proof of part (i).

Part (ii) follows readily from the disintegration formula (3.8). Indeed, let \mathcal{N} denote the Poisson process on the right-hand side of (5.2) and let $\tilde{\nu}$ be its intensity. To prove (5.2), it is enough to show that $\nu = \tilde{\nu}$.

Let $T(x) := (f(x), \theta(x))$ and consider the rectangle sets $A_{r,B} = T^{-1}((r, \infty) \times B)$ for $r > 0$ and measurable $B \subset S$. Since the class of such rectangle sets forms a π -system that generates the σ -algebra on $\mathbb{R}^d \setminus D$, it is enough to show that $\nu(A_{r,B}) = \tilde{\nu}(A_{r,B})$, for all $r > 0$, and measurable $B \subset S$.

Since f is 1-homogeneous and $f(\Theta_i) = 1$, we have that

$$T(c^{1/\alpha} \Gamma_i^{-1/\alpha} \Theta_i) = (c^{1/\alpha} \Gamma_i^{-1/\alpha}, \Theta_i).$$

Therefore, for all $r > 0$ and measurable $B \subset S$, for $A_{r,B} = T^{-1}((r, \infty) \times B)$, we have

$$\begin{aligned}
 P(\tilde{\mathcal{N}} \cap A_{r,B} = \emptyset) &= P\left(\left\{i \in \mathbb{N} : c^{1/\alpha} \Gamma_i^{-1/\alpha} \in (r, \infty), \Theta_i \in B\right\} = \emptyset\right) \\
 (5.4) \qquad &= \sum_{n=0}^{\infty} P(\Theta_1 \in B^c)^n P(|\Pi \cap (r, \infty)| = n),
 \end{aligned}$$

where Π denotes the Poisson process $\{c^{1/\alpha} \Gamma_i^{-1/\alpha}, i \in \mathbb{N}\}$. The latter equals

$$\begin{aligned}
 \sum_{n=0}^{\infty} P(\Theta_1 \in B^c)^n \frac{(cr^{-\alpha})^n}{n!} e^{-cr^{-\alpha}} &= e^{-cr^{-\alpha}} e^{cr^{-\alpha} P(\Theta_1 \in B^c)} \\
 e^{-cr^{-\alpha} P(\Theta_1 \in B)} &= \exp\left\{-c \int_r^{\infty} \frac{\alpha d\tau}{\tau^{\alpha+1}} \frac{1}{c} \sigma_S(B)\right\} \\
 (5.5) \qquad &= \exp\left\{-\int_0^{\infty} \int_S 1_{A_{r,B}}(\tau\theta) \frac{\alpha d\tau}{\tau^{\alpha+1}} \sigma_S(d\theta)\right\}.
 \end{aligned}$$

Since $P(\tilde{\mathcal{N}} \cap A_{r,B} = \emptyset) = \exp\{-\tilde{\nu}(A_{r,B})\}$, Relations (5.4) and (5.5) imply that

$$\tilde{\nu}(A_{r,B}) = \int_0^{\infty} \int_S 1_{A_{r,B}}(\tau\theta) \frac{\alpha d\tau}{\tau^{\alpha+1}} \sigma_S(d\theta).$$

This, in view of the disintegration formula (3.8), yields $\nu(A_{r,B}) = \tilde{\nu}(A_{r,B})$ and hence $\nu = \tilde{\nu}$.

We now prove part (iii). Observe that the map $T \equiv (f, \theta) : \mathbb{R}^d \setminus \{f = 0\} \rightarrow (0, \infty) \times S$ is a homeomorphism. Therefore, we can equivalently view the convergence in (5.1) in polar coordinates. More precisely, by letting $F_{n,i} := f(a_n^{-1} X_i)$ and $\Theta_{n,i} := \theta(a_n^{-1} X_i)$, the continuous mapping theorem applied to (5.1), yields

$$(5.6) \qquad T(\mathcal{N}_n \cap \{f > 0\}) \equiv \{(F_{n,i}, \Theta_{n,i}), i = 1, \dots, n\} \cap (0, \infty) \times S \implies \left\{(c^{1/\alpha} \Gamma_i^{-1/\alpha}, \Theta_i), i \in \mathbb{N}\right\},$$

as $n \rightarrow \infty$. Note that $T(c^{1/\alpha} \Gamma_i^{-1/\alpha} \Theta_i) = (c^{1/\alpha} \Gamma_i^{-1/\alpha}, \Theta_i)$.

Now, given a point measure $\Pi_n := \{(f_i, \theta_i), i = 1, \dots, n\}$ in $\mathbb{R}^d \setminus \{f = 0\}$, introduce the *order statistics* map:

$$G_m(\Pi_n) := \left((f_{k(1;n)}, \theta_{k(1;n)}), \dots, (f_{k(m;n)}, \theta_{k(m;n)})\right),$$

where $f_{k(1;n)} \geq f_{k(2;n)} \geq \dots \geq f_{k(m;n)}$ are the top order statistics of the sample f_i , $i = 1, \dots, n$ with ties resolved by taking the indices in an increasing order. If $n < m$, we formally let $G_m(\Pi_n) = ((1, \theta_0), \dots, (1, \theta_0))$ for some fixed $\theta_0 \in S$.

Let $M_p(\mathbb{R}^d \setminus \{f = 0\})$ denote the space of locally finite point measures equipped with the vague convergence topology. It is easy to show that the so-defined map $G_m : M_p(\mathbb{R}^d \setminus \{f = 0\}) \rightarrow \left((0, \infty) \times S\right)^m$ is continuous on the range of the Poisson point process $T(\mathcal{N}) = \{(c^{1/\alpha} \Gamma_i^{-1/\alpha}, \Theta_i), i \in \mathbb{N}\}$. This is because there are no ties among the Γ_i 's (with probability one) and moreover

$$G_m(\mathcal{N}) = \{(c^{1/\alpha} \Gamma_i^{-1/\alpha}, \Theta_i), i = 1, \dots, m\}.$$

The continuous mapping theorem applied to (5.6) then implies $G_m(\mathcal{N}_n) \Rightarrow G_m(\mathcal{N})$, as $n \rightarrow \infty$. Since, as $n \rightarrow \infty$, with probability converging to one, at least m of the losses $f(X_i)$, $i = 1, \dots, n$ are positive, we have

$$P\left(G_m(\mathcal{N}_n) = \left\{ \left(f(a_n^{-1} X_{k(i;n)}), \theta(X_{k(i;n)}) \right), i = 1, \dots, m \right\} \right) \longrightarrow 1, \quad \text{as } n \rightarrow \infty.$$

This implies

$$(5.7) \quad \left\{ \left(f(a_n^{-1} X_{k(i;n)}), \theta(X_{k(i;n)}) \right), i = 1, \dots, m \right\} \\ \implies \{(c^{1/\alpha} \Gamma_i^{-1/\alpha}, \Theta_i), i = 1, \dots, m\},$$

as $n \rightarrow \infty$, where the last convergence is in the sense of weak convergence of probability distributions on $((0, \infty) \times S)^m$. Another application of the continuous mapping theorem to (5.7) with the map T^{-1} applied component-wise yields (5.3) and the proof is complete. \square

6. EXAMPLES

Let $f : \mathbb{R}^d \rightarrow [0, \infty)$ be a continuous 1-homogeneous function. Suppose also that f *extends* to a continuous function $f : \overline{\mathbb{R}}^d \rightarrow [0, \infty]$. This is a non-trivial requirement as shown in Remark 3.16. Letting $D := \{f = 0\}$, we then obtain that $\tau := f$ and $\theta_f(x) := x/f(x)$ can serve as *polar coordinates* in $\overline{\mathbb{R}}^d \setminus D$. This, since $f(\theta) = 1$, simplifies the stochastic representation in Proposition 3.18 to

$$Y \stackrel{d}{=} C^{1/\alpha} Z \Theta, \quad \text{with } C = \sigma_S(S),$$

where $\Theta \sim \sigma_S(\cdot)/\sigma_S(S)$. In particular, if $C = \nu\{f > 1\} = \sigma_S(S) = 1$, we obtain

$$Y \stackrel{d}{=} Z \Theta.$$

These laws will be referred to as *standard* f -implicit max-stable. They are obtained by simply rescaling an f -implicit max-stable vector with the constant $\nu\{f > 1\}^{1/\alpha}$.

Example 6.1 (Pareto–Dirichlet implicit max-stable laws). Consider Example 3.6 where $X = (1/U_i^{1/\alpha_i})_{i=1}^d$ is a vector of independent standard α_i –Pareto components. As shown therein, we have $X \in RV_\alpha(\{n^{1/\alpha}\}, D, \nu)$, where $\alpha = \sum_{i=1}^d$. Let, as in that example,

$$f(x_1, \dots, x_d) = \left(\sum_{i=1}^d \frac{1}{x_i} \right)^{-1}.$$

Proposition 3.18 and the representation of the spectral measure imply that the *standard* (f, ν) -implicit max-stable vector W has the following stochastic representation:

$$(6.1) \quad W = Z \Theta \stackrel{d}{=} Z/\xi,$$

where $\xi = (\xi_1, \dots, \xi_d)$ has the Dirichlet($\alpha_1, \dots, \alpha_d$) distribution. We shall refer to W in (6.1) as to a *Pareto–Dirichlet implicit max-stable* distribution.

This discussion suggests that that any other (f, ν) -implicit max-stable law, with different homogeneous function f can be represented by *tilting* W in (6.1).

Proposition 6.2. *Let ν be as in (3.9) and let Y be (f, ν) -implicit max-stable. Then, for all bounded measurable function h , we have*

$$(6.2) \quad \mathbb{E}h(Y) = c^{-\alpha} \mathbb{E} \left[h(cZ\Theta/f(\Theta)) f^\alpha(\Theta) \right],$$

where $c^\alpha = \mathbb{E}f^\alpha(\Theta)$, and where Z and $\xi := 1/\Theta$ are independent standard α -Fréchet and Dirichlet($\alpha_1, \dots, \alpha_d$), respectively.

The proof is an immediate consequence of Relation (6.1) and Proposition 3.18. This result shows a type of change of measure representation for general implicit max-stable laws that are regularly varying with exponent measure ν . Unlike the classical case, where the spectral measure of a multivariate (explicit) max-stable law completely determines the distribution up to a scaling factor. The implicit max-stable laws depend in a non-trivial way on the underlying function f .

Remark 6.3. Proposition 6.2 can be used to efficiently simulate functionals of (f, ν) -implicit max-stable distributions using *importance sampling*.

Example 6.4 (Classic regular variation). Let $X = (X^{(i)})_{i=1}^d \in RV_\alpha(\{a_n\}, \{0\}, \nu)$, i.e., we have regularly variation in the usual cone $\mathbb{R}^d \setminus \{0\}$. In this case, we also have $X \in RV_\alpha(\{a_n\}, D, \nu|_{\mathbb{R}_D^d})$, for any cone \mathbb{R}_D^d such that $\nu(\mathbb{R}_D^d) > 0$. Thus, for any continuous homogeneous loss function f such that $\nu(\{f > 0\}) > 0$, by Theorem 3.14, the implicit maxima of independent copies of X converge to a non-trivial (f, ν) -implicit max-stable law. The structure of these distributions depends on the loss and the spectral measure σ of ν . They can be readily expressed as shown in Proposition 3.18. Ultimately, a variety of implicit max-stable models, tailored to specific losses and applications can be developed. This is beyond the scope of the present work.

In this example, we discuss the case of elliptical losses. Since $X \in RV_\alpha(\{a_n\}, \{0\}, \nu)$, in this case, *any* norm \mathbb{R}^d leads to valid polar coordinates. Let for example $\tau(x) := \|x\|_2$ be the Euclidean norm. Then, by standard $(\|\cdot\|_2, \nu)$ -implicit max-stable law has the following representation

$$W = Z\Theta,$$

where Θ has distribution $\sigma_0(\cdot) := \nu(\theta^{-1}(\cdot) \cap \{\tau > 1\})/\nu(\{\tau > 1\})$ on the unit Euclidean sphere S .

By complete analogy with Proposition 6.2, Relation (6.2) holds, for any (f, ν) -implicit max-stable vector Y . This allows us to simulate Y through tilting. For example, one can determine the structure of all such laws where f has *elliptical* contours. That is, suppose

$$f(x) = \psi(x^\top \Sigma x) =: \psi(\|x\|_\Sigma^2),$$

where Σ is a symmetric positive definite matrix and $\psi : [0, \infty) \rightarrow \mathbb{R}$ is strictly monotone. By Remark 3.19, Theorem 3.14 applies to the continuous and 1-homogeneous function $(\psi^{-1} \circ f)^{1/2}(x) = \|x\|_\Sigma^{1/2}$. Therefore the implicit extreme value laws Y corresponding to f (equivalently, the (f, ν) -implicit max-stable ones) have the representation

$$\mathbb{E}h(Y) = c^{-\alpha} \mathbb{E} \left[h(cZ\Theta/\|\Theta\|_\Sigma) \|\Theta\|_\Sigma^\alpha \right],$$

where $c^\alpha = \mathbb{E}\|\Theta\|_\Sigma^\alpha$ and where $\Theta \sim \sigma_0$.

Note that the distribution of Y , as expected, does not depend on ψ . Suppose for example that $1/\psi(x) \propto \phi_\Sigma(x)$ is the density of centered multivariate Normal distribution with covariance matrix $2\Sigma^{-1}$. Suppose also that X_i are as in Theorem 3.14 and let

$$k(n) := \underset{i=1, \dots, n}{\operatorname{Argmax}} f(X_i) = \underset{i=1, \dots, n}{\operatorname{Argmin}} \phi_\Sigma(x).$$

In this case, the limit distribution of $a_n^{-1}X_{k(n)}$ describes the large-sample behavior of *novelties* relative to the Gaussian model $\phi_\Sigma(x)$ in the sense of Clifton *et al* [2].

Example 6.5 (Gaussian copula). Let $Z = (Z_1, Z_2)^\top$ be bivariate Normal random vector having standard Normal margins and correlation $\rho = \mathbb{E}(Z_1 Z_2) \in (-1, 1)$. Let $\overline{\Phi}(z) = P(Z_1 > z)$, $z \in \mathbb{R}$ denote the complementary cdf of Z_1 . Consider the random vector

$$X = \left(\frac{1}{\overline{\Phi}(Z_1)}, \frac{1}{\overline{\Phi}(Z_2)} \right)^\top.$$

Observe that X has standard unit Pareto marginals and its dependence is determined by the Gaussian copula. It is well known that the components of X are asymptotically independent, or equivalently that $X \in RV_1(\{0\}, \nu)$, where the measure ν concentrates on the two positive axes. If one excises the axes and considers regular variation in $(0, \infty)^2$, however, a finer *hidden regular variation* emerges. More precisely, letting $\mathbb{R}^2 \setminus D = (0, \infty)^2$, by Example 2.1 (p. 255) in Draisma *et al* [7], we have that

$$X \in RV_\alpha(\{a_n\}, D, \nu), \quad \text{where } \alpha = \frac{2}{1 + \rho}$$

and for all $(x_1, x_2) \in (0, \infty)^2$,

$$\nu((x_1, \infty) \times (x_2, \infty)) = (x_1 x_2)^{-1/(1+\rho)}.$$

This shows that the random vector X has the same regular variation behavior as in Example 6.1 with $d = 2$ and $\alpha_1 = \alpha_2 = 1/(1+\rho)$. Therefore, with a homogeneous function $f(x_1, x_2) = (1/x_1 + 1/x_2)^{-1}$, for example, the (f, ν) -implicit max-stable distribution attracting X is of the form (6.1). All results for Pareto–Dirichlet laws above apply in this particular setting.

Remark 6.6. The derivation of the regular variation behavior on $(0, \infty)^d$ with $d = 2$ for Gaussian copula with Pareto margins is rather technical. To the best of our knowledge, the d -dimensional case $d > 3$ remains open.

Remark 6.7. The study of finer behavior of asymptotically independent variables was initiated with the seminal work of Ledford and Tawn [12] (see also [9, 10, 8, 5] among others).

7. ACKNOWLEDGEMENTS

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APPENDIX A. SOME PROOFS AND AUXILIARY LEMMAS

Proof of Fact 3.1. All open sets in $\overline{\mathbb{R}}_D^d$ are precisely of the type $V \setminus D$, where V is open in $\overline{\mathbb{R}}^d$.

(\Leftarrow) Let $U_n := V_n \setminus D$, $n \in \mathbb{N}$ be an open cover of F in $\overline{\mathbb{R}}_D^d$, where the V_n -s are open in $\overline{\mathbb{R}}^d$. Observe that since F is closed in $\overline{\mathbb{R}}_D^d$, then $F = K \setminus D$, for some K that is closed in $\overline{\mathbb{R}}^d$. Since D is also closed, $F \cup D = K \cup D$ is closed and hence compact in $\overline{\mathbb{R}}^d$. Now, the fact that the open set U covers D , implies that $\{U, V_n, n \in \mathbb{N}\}$ is an open cover of the compact $K \cup D$ in $\overline{\mathbb{R}}^d$. Thus, there exists a finite N , such that

$$F \cup D \equiv K \cup D \subset U \cup \bigcup_{n=1}^N V_n.$$

This, since $F \cap U = \emptyset$ implies that $F \subset \bigcup_{n=1}^N V_n \setminus D \equiv \bigcup_{n=1}^N U_n$, which is a finite sub-cover of F in $\overline{\mathbb{R}}_D^d$, showing that F is compact in $\overline{\mathbb{R}}_D^d$.

(\Rightarrow) For all $\epsilon > 0$, let $D_\epsilon := \{x \in \overline{\mathbb{R}}^d : \rho(x, D) \leq \epsilon\}$, where $\rho(x, D) = \min_{y \in D} \rho(x, y)$ is the distance from x to the compact D in $\overline{\mathbb{R}}^d$ with ρ as in (3.1). Note that the sets D_ϵ ($\epsilon > 0$) are closed and $D = \bigcap_{\epsilon > 0} D_\epsilon$. Let $U_\epsilon := \overline{\mathbb{R}}^d \setminus D_\epsilon$ and observe that $\{U_\epsilon, \epsilon > 0\}$ is an open cover of F in $\overline{\mathbb{R}}_D^d$. Since F is compact, it is also covered by a finite subset of U_ϵ -s. Since the latter are nested, it follows that $F \subset U_{\epsilon_0}$, for some ϵ_0 . By taking $U := \{x \in \overline{\mathbb{R}}^d : \rho(x, D) < \epsilon_0\}$, we obtain that $F \cap U$ and $D \subset U$, which shows that F is bounded away from D . \square

Proof of Proposition 3.9. (\Rightarrow) The continuity and homogeneity of τ imply that $\partial\{\tau > t\} = \{\tau = t\} = t\{\tau = 1\}$, $t > 0$. Since $\nu\{\tau > \epsilon\} < \infty$ for any $\epsilon > 0$ and since the set $\{\tau > \epsilon\}$ equals the disjoint union $\bigcup_{t > \epsilon} \{\tau = t\} = \bigcup_{t > \epsilon} t\{\tau > 1\}$, we obtain that $\nu(\{\tau = t\}) = 0$, $\forall t > 0$, i.e., $\{\tau > t\}$ is a ν -continuity set for all $t > 0$. Thus, in view of the homogeneity of τ and the scaling property of ν , Relation (3.2) implies that, for all $x > 0$, as $n \rightarrow \infty$,

$$(A.1) \quad nP(a_n^{-1}\tau(X) > x) = nP(a_n^{-1}X \in \{\tau > x\}) \longrightarrow \nu(\{\tau > x\}) \equiv \nu(\{\tau > 1\})x^{-\alpha}.$$

We have moreover that the function $u \mapsto P(\tau(X) > u)$ varies regularly, with exponent $(-\alpha)$. This follows from its monotonicity and Theorem 1.10.3 on p. 55 in [1].

Introduce now the probability measures $Q_u(\cdot) := P(\theta(X) \in \cdot | \tau(X) > u)$, $u > 0$ on $(\overline{S}, \mathcal{B}(\overline{S}))$, where the measure is extended as zero on the infinite points in $\overline{S} \setminus S$, with $\overline{S} := \{\tau = 1\}$. We will first show that $Q_{a_n} \xrightarrow{w} \sigma_0$, for some probability measure σ_0 . Indeed, by (A.1), as $n \rightarrow \infty$

$$\begin{aligned} Q_{a_n}(B) &= \frac{P(\theta(X) \in B, \tau(X) > a_n)}{P(\tau(X) > a_n)} \\ &\sim \frac{nP(a_n^{-1}X \in T^{-1}((1, \infty] \times B))}{\nu\{\tau > 1\}} := \mu_n(T^{-1}((1, \infty] \times B)), \end{aligned}$$

where $T(x) := (\tau(x), \theta(x))$. Recall that $T : \overline{\mathbb{R}}_D^d \rightarrow (0, \infty) \times \overline{S}$ is a homeomorphism (recall (3.5) and the discussion thereafter). Therefore, for every open set $B \subset \overline{S}$ (in the relative topology), the set $A := T^{-1}((1, \infty) \times B)$ is open. Further, since $A \subset \{\tau \geq 1\}$, where $\{\tau \geq 1\}$

is compact (in $\overline{\mathbb{R}}_D^d$), the vague convergence in (3.2) coincides with the weak convergence of finite measures restricted to the set $\{\tau \geq 1\}$. Hence, by the Portmanteau characterization of weak convergence (see e.g. Theorem A 2.3.II on p. 391 in [3]), Relation (3.2) implies that, for all open sets $B \subset \overline{S}$,

$$\liminf_{n \rightarrow \infty} Q_{a_n}(B) = \liminf_{n \rightarrow \infty} \mu_n(A) \geq \frac{\nu(A)}{\nu\{\tau > 1\}} =: \sigma_0(B),$$

Since the last relation is valid for all open sets $B \subset \overline{S}$, the Portmanteau theorem applied to the probability measures Q_{a_n} , shows that $Q_{a_n} \xrightarrow{w} \sigma_0$, as $n \rightarrow \infty$.

Let now $u_n \rightarrow \infty$ be arbitrary. We will show that $Q_{u_n} \xrightarrow{w} \sigma_0$. Let $a_n^* := \inf_{m \geq n} a_m$ and note that by Theorem 1.5.3 on p. 23 of [1], we have $a_n^* \sim a_n$, $n \rightarrow \infty$. Further, the fact that $u \mapsto P(\tau(X) > u)$ is regularly varying at infinity, implies that $P(\tau(X) > a_n^*) \sim P(\tau(X) > a_n)$ since the convergence $nP(\tau(X) > a_n x) \rightarrow Cx^{-\alpha}$ is uniform in x on each fixed interval $[c, \infty)$, $c > 0$ (cf Theorem 1.5.2 in [1]). Since $a_n^* \uparrow \infty$, there exists an integer sequence $k_n \rightarrow \infty$, such that for all sufficiently large n ,

$$a_{k_n}^* \leq u_n < a_{k_n+1}^*.$$

Hence, for any measurable $B \subset S$, we have

$$(A.2) \quad \frac{P(\theta(X) \in B, \tau(X) \geq a_{k_n}^*)}{P(\tau(X) > a_{k_n}^*)} < Q_{u_n}(B) \leq \frac{P(\theta(X) \in B, \tau(X) \geq a_{k_n+1}^*)}{P(\tau(X) > a_{k_n+1}^*)}.$$

By the fact that $mP(\tau(X) > a_m^*) \rightarrow C > 0$, we get $P(\tau(X) > a_{k_n+1}^*) \sim P(\tau(X) > a_{k_n}^*)$. This, since $Q_{a_n^*} \xrightarrow{w} \sigma_0$, shows that the upper and lower bounds of $Q_{u_n}(B)$ in (A.2) converge to $\sigma_0(B)$, for all continuity sets B , which completes the proof of the ‘only if’ part. The fact that ν does not put charge on the infinite points implies that σ_0 concentrates on $S \equiv \overline{S} \cap \mathbb{R}^d$.

(\Leftarrow) Suppose now that (A.1) holds. Consider the semiring of subsets of $\overline{\mathbb{R}}_D^d$,

$$\mathcal{R} := \{T^{-1}((x, y] \times B) : 0 < x < y < \infty, B \in \mathcal{B}(S)\},$$

where $T(x) = (\tau, \theta)$ and $\mathcal{B}(S)$ is the class of Borel measurable sets in S . Define the σ -finite measure

$$\nu(T^{-1}((x, \infty) \times B)) := Cx^{-\alpha}\sigma_0(B), \quad (x, \infty] \times B \in \mathcal{R}.$$

Since \mathcal{R} is a π -system generating the Borel σ -algebra $\mathcal{B}(\overline{\mathbb{R}}_D^d)$, the mapping ν uniquely extends to a σ -finite measure on $(\mathbb{R}_D^d, \mathcal{B}(\mathbb{R}_D^d))$. We further extend ν to $(\overline{\mathbb{R}}_D^d, \mathcal{B}(\mathbb{R}_D^d))$ by defining as zero at infinity.

By (3.12), we readily obtain that for all $A = T^{-1}((x, \infty) \times B)$, where $x > 0$ and where $B \in \mathcal{B}(S)$ is a continuity set of σ_0 , that

$$nP(a_n^{-1}X \in A) = P(\theta(X) \in B | a_n^{-1}\tau(X) > x)P(a_n^{-1}\tau(X) > x) \longrightarrow \nu(A),$$

as $n \rightarrow \infty$. Thus, we also have that $nP(a_n^{-1}X \in A) \rightarrow \nu(A)$, $n \rightarrow \infty$, for all sets in the semi ring \mathcal{R} such that $\theta(A)$ is a σ_0 -continuity set.

To prove (3.2), since ν is supported on \mathbb{R}_D^d , it is enough to show weak convergence of the measures restricted to $\{\tau \in (\epsilon, \infty)\}$, for each $\epsilon > 0$. Note however that the restriction of \mathcal{R} to $\{\tau \in (\epsilon, \infty)\}$ is a covering semi ring for the separable metric space $\{\tau \in (\epsilon, \infty)\}$.

Therefore, \mathcal{R} is a convergence determining class (cf Proposition A 2.3.IV on p. 393 in [3]) and the already established weak convergence for \mathcal{R} implies the result. \square

The following result was used in the proof of Theorem 3.14. Consider a locally compact metric space (E, ρ) with countable base equipped with its Borel σ -algebra. More precisely, we shall assume that all closed and ρ -bounded sets in E are compact. Recall that a set $A \subset E$ is ρ -bounded, if A is contained in a ball $B(x, r) = \{y \in E : \rho(x, y) < r\}$, for some $x \in E$ and $r > 0$. Borel measures that are finite on all compacts are referred to as Radon.

Lemma A.1. *Let ν_n and ν be Radon measures on E . Suppose that h_n and h are locally bounded and non-negative measurable functions defined on E . Introduce the Radon measures*

$$\mu_n(A) := \int_A h_n(x) \nu_n(dx) \quad \text{and} \quad \mu(A) := \int_A h(x) \nu(dx), \quad A \in \mathcal{B}(E).$$

Suppose that $\nu(\overline{\text{Disc}(h)}) = 0$ and that for every compact $K \subset E$, we have that $\sup_{x \in K} |h_n(x) - h(x)| \rightarrow 0$, $n \rightarrow \infty$, that is, h_n converges to h locally uniformly.

Then, the convergence $\nu_n \xrightarrow{v} \nu$, as $n \rightarrow \infty$ implies $\mu_n \xrightarrow{v} \mu$, as $n \rightarrow \infty$. Furthermore, if μ_n are probability measures, then $\mu(E) \leq 1$.

Proof. We need to show that for all continuous functions g with compact support, we have $\int_E g d\mu_n \rightarrow \int_E g d\mu$, as $n \rightarrow \infty$, or equivalently, $\int_E g h_n d\nu_n \rightarrow \int_E g h d\nu$, as $n \rightarrow \infty$. By the triangle inequality, we have that

$$(A.3) \quad \left| \int_E g h_n d\nu_n - \int_E g h d\nu \right| \leq \int_E |g(h_n - h)| d\nu_n + \left| \int_E g h d\nu_n - \int_E g h d\nu \right| =: I_n + J_n.$$

For the term I_n , for all $\delta > 0$, we have

$$(A.4) \quad I_n \leq C \sup_{x \in K} |h_n(x) - h(x)| \nu_n(K) \leq C \sup_{x \in K} |h_n(x) - h(x)| \nu_n(K^\delta),$$

where $C = \sup_{x \in E} |g(x)|$, the compact support of g is denoted by K , and $K^\delta := \{y \in E : \rho(x, y) \leq \delta\}$ is its closed δ -neighborhood. Note that the closed and bounded set K^δ is compact and hence $\nu(K^\delta) < \infty$. Also, K^{δ_1} is contained in the interior of K^{δ_2} , for all $0 < \delta_1 < \delta_2$ and hence $\partial(K^\delta)$ are disjoint for all $\delta > 0$. This implies that $\nu(\partial K^\delta) = 0$, for all but countably many $\delta > 0$, since the Radon measure ν is σ -finite and has at most countably many atoms. Thus, K^δ is a continuity set of ν , for some $\delta > 0$. This yields $\lim_{n \rightarrow \infty} \nu_n(K^\delta) = \nu(K^\delta) < \infty$, and the right-hand side of (A.4) vanishes, as $n \rightarrow \infty$.

Now, we focus on J_n in (A.3). Let $\delta > 0$ and define

$$(A.5) \quad h_\delta(x) := h(x) \tau_\delta(x), \quad \text{where } \tau_\delta(x) := \exp \left\{ \frac{1}{\delta} - \frac{1}{\delta \wedge \rho(x, F)} \right\},$$

with $F := \overline{\text{Disc}(h)} \cap K$. The function τ_δ is continuous, $|\tau_\delta| \leq 1$ and it vanishes on the set F . Further, $\tau_\delta(x) = 1$, if $\rho(x, F) \geq \delta$. By the triangle inequality, we have

$$J_n \leq \int_E |g(h - h_\delta)| d\nu_n + \left| \int_E g h_\delta (d\nu_n - d\nu) \right| + \int_E |g(h - h_\delta)| d\nu =: J_{n,1} + J_{n,2} + J_{n,3}.$$

Note that for each fixed $\delta > 0$, the function gh_δ is continuous and has compact support. Therefore, the vague convergence $\nu_n \xrightarrow{v} \nu$, $n \rightarrow \infty$ implies $J_{n,2} \rightarrow 0$, $n \rightarrow \infty$. Now, by the local boundedness of h and the fact that $\{h \neq h_\delta\} \subset F^\delta$, we have

$$J_{n,1} + J_{n,3} \leq C\nu_n(F^\delta) + C\nu(F^\delta),$$

where $C = \sup_{x \in K} |g(x)h(x)| < \infty$ and $F^\delta = \{\tau_\delta < 1\}$. As argued above, for all $0 < \delta_1 < \delta_2$, the set F^{δ_1} is contained in the interior of F^{δ_2} . Thus, the σ -finiteness of ν implies that the F^δ s are ν -continuity sets, for all but countably many $\delta > 0$. Further, we have $\nu(F^\delta) \downarrow \nu(F) = 0$, as $\delta \downarrow 0$ since $\nu(F^\delta) < \infty$ and $F^\delta \downarrow F$, $\delta \downarrow 0$. Thus, for every $\epsilon > 0$, we can pick $\delta > 0$ such that F^δ is a ν -continuity set and $\nu(F^\delta) < \epsilon/(3C)$. This ensures that $J_{n,3} < \epsilon/3$ and $\nu_n(F^\delta) \rightarrow \nu(F^\delta)$, $n \rightarrow \infty$, so that $J_{n,1} < \epsilon/2$, for all sufficiently large n . This, since $\epsilon > 0$ was arbitrary yields $J_{n,1} + J_{n,2} \rightarrow 0$, $n \rightarrow \infty$, which completes the proof. \square

Lemma A.2. *Let X and f be as in Assumptions $RV_\alpha(D, \nu)$ and H . If in addition, f and ν satisfy Assumptions F and C , then*

(i) *The sets $A_u := \{f > u\} \equiv u\{f > 1\}$, $u > 0$ are ν -continuity sets for all but countably many u -s.*

(ii) *With a_n as in (3.2), for all $y > 0$, we have*

$$(A.6) \quad nP(f(X) > a_n y) \longrightarrow \nu(\{f > 1\})y^{-\alpha}, \quad \text{as } n \rightarrow \infty.$$

(iii) *The function $y \mapsto P(f(X) > y)$ is regularly varying of exponent $-\alpha$ and hence the convergence in (A.6) is uniform in y on $[c, \infty)$, for any fixed $c > 0$.*

Proof. (i): We need to show that $\nu(\partial A_u) = \nu(\overline{A_u} \setminus \langle A_u \rangle) = 0$, for all but countably many u -s. If f is continuous, then $\partial A_u = \{f = u\}$ and these sets are disjoint for all $u > 0$. Hence, for all $\epsilon > 0$, $\nu(A_\epsilon) = \cup_{u > \epsilon} \partial A_u$, which is finite by Assumption F. This shows that A_u -s are ν -continuity sets for all but countably many $u > 0$. When f is discontinuous, however, $\partial\{f > u\} \neq \{f = u\}$ and this argument fails. The intuition is that jumps in the angular component in (3.18) can lead to non-trivial overlaps between the boundaries ∂A_u for entire ranges of u -s. The role of Assumption C is to make such overlaps negligible in ν -measure. We shall now make this precise.

Fix an arbitrary $\epsilon > 0$. By Assumption F, there exists an open set $U \subset \overline{\mathbb{R}^d}$, such that $D \subset U$ and $\{f > \epsilon\} \subset U^c := \overline{\mathbb{R}^d} \setminus U$. Since U^c is closed, we also have that $\overline{\{f > \epsilon\}} \subset U^c$. Note that the two disjoint sets D and U^c are compact in $\overline{\mathbb{R}^d}$ and hence they can be separated. Namely,

$$\rho(D, U^c) := \inf_{x \in D, y \in U^c} \rho(x, y) =: \epsilon_0 > 0,$$

where ρ is the metric in (3.1). Consider the open $\epsilon_0/2$ -neighborhood of D :

$$D_{\epsilon_0/2} = \{x \in \overline{\mathbb{R}^d} : \rho(x, D) < \epsilon_0/2\}$$

and define the following compact in $\overline{\mathbb{R}^d} \setminus D$

$$(A.7) \quad F := \overline{\text{Disc}(f)} \cap D_{\epsilon_0/2}^c.$$

Observe that since $\{f > \epsilon\} \subset U^c \subset D_{\epsilon_0/2}^c$, we have

$$(A.8) \quad \text{Disc}(f) \cap \{f > \epsilon\} \subset \overline{\text{Disc}(f)} \cap U^c \subset F \subset \overline{\text{Disc}(f)}.$$

In particular, $\nu(F) = 0$, because of Assumption C.

The intuition behind the set F is that it collects all discontinuity points of f over the region $\{f > \epsilon\}$, and therefore over the regions $\{f > u\}$, for $u > \epsilon$. We shall regularize f and replace it by a function that is continuous on $D_{\epsilon_0/2}^c \supset \overline{\{f > \epsilon\}}$ and coincides with f , except for a small neighborhood of F . Letting the neighborhood shrink, we will arrive at the desired claim. Now, the details.

For each $\delta \in (0, \epsilon_0/2)$, define the function τ_δ as in (A.5). Note that

$$F_\delta := \{x \in \mathbb{R}^d : \rho(x, F) < \delta\} = \{\tau_\delta < 1\}$$

is the open δ -neighborhood of F . Since F is bounded away from 0, so are F_δ for all sufficiently small $\delta > 0$. Thus, $\nu(F_\delta) < \infty$, eventually, as $\delta \downarrow 0$, and hence $\nu(F_\delta) \downarrow \nu(\cap_{\delta>0} F_\delta) \equiv \nu(F) = 0$.

We are now ready to study $\nu(\partial\{f > u\})$ for $u > \epsilon$. Define the functions $f_\delta(x) := f(x)\tau_\delta(x)$, $\delta \in (0, \epsilon_0/2)$. By construction, $f(x) = f_\delta(x)$ for all $x \in F_\delta^c$ and thus,

$$\{f > u\} = \left(\{f_\delta > u\} \cap F_\delta^c\right) \cup \left(\{f > u\} \cap F_\delta\right),$$

which implies

$$\partial\{f > u\} \subset \partial\left(\{f_\delta > u\} \cap F_\delta^c\right) \cup \partial\left(\{f > u\} \cap F_\delta\right).$$

Now, using the facts that $\partial(A \cap B) \subset \partial A \cup \partial B$ and $\partial(F_\delta^c) = \partial F_\delta \subset \overline{F_\delta}$, we obtain

$$(A.9) \quad \partial\{f > u\} \subset \partial\{f_\delta > u\} \cup \overline{F_\delta}.$$

We will show next that $\nu(\partial\{f_\delta > u\}) = 0$, for all but countably many $u > \epsilon$. Since $\tau_\delta \leq 1$, we have $f_\delta \leq f$ and thus

$$\overline{\{f_\delta > u\}} \subset \overline{\{f > \epsilon\}}, \quad \text{for all } u > \epsilon.$$

This, in view of (A.8), implies that $\overline{\{f_\delta > u\}} \subset D_{\epsilon_0/2}^c$, for all $u > \epsilon$. By the construction of the set F in (A.7), however, the function f_δ is continuous on $D_{\epsilon_0/2}^c$, and hence $\partial\{f_\delta > u\} = \{f_\delta = u\}$, for all $u > \epsilon$. Thus, as argued above, the fact that $\nu(\{f_\delta > \epsilon\}) < \infty$ implies $\nu(\partial\{f_\delta > u\}) = 0$ for all but countably many $u > \epsilon$. (The set of u -s may depend on the choice of δ .)

On the other hand, $\overline{F_\delta} \subset F_{2\delta}$ and as shown $\nu(F_{2\delta}) \downarrow 0$ as $\delta \downarrow 0$. Thus, by taking a limit over a countable sequence $\delta_m \downarrow 0$, we see that the ν -measure of the left-hand side in (A.9) vanishes for all but countably many $u > \epsilon$. This completes the proof of part (i).

We now prove (ii). For all $y > 0$, by the homogeneity of f (Assumption H),

$$\begin{aligned} nP(f(X) > a_n y) &= nP(X \in f^{-1}(a_n y, \infty)) \\ &= nP(X \in a_n \{f > y\}) =: nP(X \in a_n A_y). \end{aligned}$$

Now, by the already established part (i), all but countably many A_y -s are ν -continuity sets. Thus, by (3.2),

$$(A.10) \quad nP(a_n^{-1}X \in A_y) \rightarrow \nu(A_y) \equiv \nu(\{f > 1\})y^{-\alpha}, \quad \text{as } n \rightarrow \infty,$$

for all but countably many y -s. The monotonicity (in y) of the left-hand side in (A.10) and the continuity (in y) of the limit, imply that Relation (A.6) holds for all $y > 0$. This completes the proof of part (ii).

(iii): Observe that the sequence a_n in (3.2) is regularly varying with exponent $1/\alpha$, (A.6) holds for all $y > 0$, and the function $u \mapsto P(f(X) > u)$ is monotone. Therefore, Theorem 1.10.3 on p. 55 in [1] applies and shows that $u \mapsto P(f(X) > u)$ is regularly varying, with index $-\alpha$. By Theorem 1.5.2 on p. 22 in [1] the convergence in (A.6) is also uniform in y on $[c, \infty)$, for all $c > 0$. \square

The following slight reformulation of Scheffe's Lemma is useful.

Lemma A.3 (induced Scheffe's Lemma). *Let (E, \mathcal{E}, μ) be a measure space and let $T : (E, \mathcal{E}) \rightarrow (F, \mathcal{F})$ be an $\mathcal{E}|\mathcal{F}$ -measurable mapping. Suppose that $p_n, p \in L^1(E, \mathcal{E}, \mu)$ are probability densities and define the probability measures Q_n and Q on (F, \mathcal{F}) as follows:*

$$Q_n(B) := \int_E 1_B(T(x)) p_n(x) \mu(dx) \quad \text{and} \quad Q(B) := \int_E 1_B(T(x)) p(x) \mu(dx), \quad B \in \mathcal{F}.$$

If $p_n(x) \rightarrow p(x)$, $n \rightarrow \infty$, μ -a.e., then

$$\|Q_n - Q\|_{\text{tv}} := \sup_{B \in \mathcal{F}} |Q_n(B) - Q(B)| \longrightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Proof. Observe that that

$$|Q_n(B) - Q(B)| \leq \int_E 1_B(T(x)) |p(x) - p_n(x)| \mu(dx) \leq \|p_n - p\|_{L^1(\mu)}.$$

The last bound vanishes by the classic version of Scheffe's lemma. \square

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